Math 551: Applied PDE and Complex Vars — Fall 2023

Problem Set 1 — **Assigned Weds Aug 30** — **Due Fri Sep 8**

Linear Algebra Review

-1. Homework policy: Homework is to be submitted via `www.gradescope.com` or using the Gradescope tool from the Math 551 Sakai webpage.

Please SHOW ALL WORK leading up to your solutions – intermediate steps are important (and will get you partial credit)!
Unexcused late homeworks will not be accepted. Any extensions or excuses must be requested before the due date.
Office hours: We will vote (web poll) for times for regular office hours. You can always email me at any time with your questions or to request to schedule a time to meet with me.

0. Reading: Haberman, Appendix to Section 5.5 (pp. 178–183).

1. The eigen-expansion method for solving systems of linear equations

Consider the matrix equation $\mathbf{Lu} = \mathbf{b}$:

$$\begin{pmatrix} 6 & 2 & -8 \\ 18 & -19 & 36 \\ 4 & -7 & 18 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 32 \\ 1 \\ -17 \end{pmatrix}.$$

(a) You are given that two eigenvalues for $\mathbf{L}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$ are $\lambda_1 = -10, \lambda_2 = 5$. What is λ_3?

(b) Find the eigenvectors $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3\}$.[1] Scale them so that the last entry of each vector is equal to one.

(c) Show that $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3\}$ are not orthogonal, but they are linearly independent.
Hint: To test for orthogonality, calculate the dot products $\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j$.
For linear independence, recall the definition: vectors are linearly independent if $c_1\boldsymbol{\phi}_1 + c_2\boldsymbol{\phi}_2 + c_3\boldsymbol{\phi}_3 = \mathbf{0}$ only for $c_1 = c_2 = c_3 = 0$. This can be written as a matrix-vector equation, $\boldsymbol{\Phi}\mathbf{c} = \mathbf{0}$, with the vectors being the columns of $\boldsymbol{\Phi}$. What does $\det(\boldsymbol{\Phi})$ tell you about the uniqueness of the solution?

(d) Find the adjoint eigenvectors $\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3\}$. Scale them so that the last entry of each vector is equal to one.

(e) Determine the expansion coefficients c_k and compute the solution $\mathbf{u} = \sum_k c_k\boldsymbol{\phi}_k$ to confirm that this agrees with $\mathbf{u} = \mathbf{L}^{-1}\mathbf{b} = (3, -1, -2)^T$.

2. Linear algebra with a different inner product[2]

(a) Haberman page 183, Problem 5.5A.3.

(b) Find the adjoint eigenvectors of $\mathbf{A}$ with respect to the regular dot product.
Verify orthogonality by calculating the inner products $\boldsymbol{\phi}_i \cdot \boldsymbol{\psi}_j$.

(c) Let $\mathbf{M} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$. Show that the book's "$\mathbf{a} \cdot \mathbf{b}$" dot product can be written as a "weighted inner product" defined by $\langle \mathbf{a}, \mathbf{b} \rangle \equiv \mathbf{a} \cdot \mathbf{Mb}$ (or $= a_1b_1\sigma_1 + a_2b_2\sigma_2$ with $\sigma_1 = m_{1,1}$ and $\sigma_2 = m_{2,2}$).

(d) Find the positive diagonal matrix $\mathbf{C}$ such that $\mathbf{M} = \mathbf{C}^2$. *(continued)*

(continued)

[1] About notation: I'm using (parentheses) for matrices and vectors, $\langle bra, ket \rangle$ for inner products and {curly braces} for sets of things.

[2] There will be many problems coming soon which will use weighted inner products.

(e) Multiply the eigenvalue equation $\mathbf{A}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$ on the left by $\mathbf{M}$ to get $\mathbf{MA}\boldsymbol{\phi} = \lambda\mathbf{M}\boldsymbol{\phi}$. Then write $\mathbf{M} = \mathbf{C}^2$ and $\boldsymbol{\phi} = \mathbf{C}^{-1}\mathbf{y}$ in this equation and re-arrange to give an eigenvalue/eigenvector equation for $\mathbf{y}$, $\mathbf{By} = \lambda\mathbf{y}$, where $\mathbf{B}$ is a real symmetric matrix, $\mathbf{B} = \mathbf{B}^{\mathbf{T}}$.
What is the matrix $\mathbf{B}$? (please express it's general form in terms of $\mathbf{A}, \mathbf{C}$ and also specific numbers for this problem)

(f) Explain why the orthogonality relation for the $\mathbf{y}$'s is $\mathbf{y}_1 \cdot \mathbf{y}_2 = 0$ and use this to justify the orthogonality of the $\boldsymbol{\phi}$'s in the weighted inner product from part (c).
Hint: Use $\mathbf{y} = \mathbf{C}\boldsymbol{\phi}$ in the dot product.

3. Linear algebra with complex-valued vectors[3]

For vectors whose entries are complex numbers ($\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$) the inner product is defined as $\langle \mathbf{x}, \mathbf{y} \rangle \equiv \mathbf{x}^T\bar{\mathbf{y}}$, where $\bar{\mathbf{y}}$ is the complex conjugate of vector $\mathbf{y}$, conjugated entry by entry in the vector.
The conjugate of a complex number, $z = a + ib$ is defined as $\overline{z} = \overline{a + ib} = a - ib$ where $i^2 = -1$. Note that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$ for all complex numbers z, w.

(a) Show that the "real inner product" ($\mathbf{x}^T\mathbf{y}$) does not satisfy the norm property for complex vectors.
Hint: What is the value of $\mathbf{x}^T\mathbf{x}$ for the vector $\mathbf{x} = (1, i)^T$?

(b) Let $\mathbf{x} = (a+ib, c+id)^T$, where a, b, c, d are real numbers. Show that the "complex inner product" is a norm, with $||\mathbf{x}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle \geq 0$.

(c) How is the value of the complex inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ related to the value of $\langle \mathbf{y}, \mathbf{x} \rangle$?

(d) If $\mathbf{A}$ is a matrix with complex-valued entries, what is the formula for the adjoint $\mathbf{A}^*$ satisfying $\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}^*\mathbf{y} \rangle$ with respect to the complex inner product? How are the adjoint eigenvalues γ_k related to the λ_k of $\mathbf{A}$?

(e) Find $\{\lambda_k, \boldsymbol{\phi}_k\}$ and $\{\gamma_k, \boldsymbol{\psi}_k\}$ for $\mathbf{A} = \begin{pmatrix} i & -1 \\ 2 & i+2 \end{pmatrix}$ and show that $\boldsymbol{\phi}_1 \perp \boldsymbol{\psi}_2$ and $\boldsymbol{\phi}_2 \perp \boldsymbol{\psi}_1$.

(f) Haberman page 183, Problem 5.5A.6.
Hint: For part (a) of this problem, consider the complex inner product of the matrix times an eigenvector against the same eigenvector, and consider what happens when you factor a constant out of the inner product (from the first vs. second factors) to end up showing that $(\lambda_k - \overline{\lambda}_k) = 0$. (This works for both complex-Hermitian matrices and real-symmetric matrices.)

4. Solution of initial value problems for matrix-vector ODE systems

Haberman page 183, Problem 5.5A.4, part (a).

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[3]Complex inner products will be used near the end of the course when we do Fourier transforms.