# Sets of Matrices All Infinite Products of Which Converge

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ABSTRACT

An infinite product $\prod_{i=1}^{\infty} \mathbf{M}_i$ of matrices converges (on the right) if $\lim_{i\to\infty} \mathbf{M}_1 \cdots \mathbf{M}_i$ exists. A set $\Sigma = \{\mathbf{A}_i : i \geqslant 1\}$ of $n \times n$ matrices is called an *RCP set* (right-convergent product set) if all infinite products with each element drawn from $\Sigma$ converge. Such sets of matrices arise in constructing self-similar objects like von Koch's snowflake curve, in various interpolation schemes, in constructing wavelets of compact support, and in studying nonhomogeneous Markov chains. This paper gives necessary conditions and also some sufficient conditions for a set $\Sigma$ to be an RCP set. These are conditions on the eigenvalues and left eigenspaces of matrices in $\Sigma$ and finite products of these matrices. Necessary and sufficient conditions are given for a finite set $\Sigma$ to be an RCP set having a limit function $\mathbf{M}_{\Sigma}(\mathbf{d}) = \prod_{i=1}^{\infty} \mathbf{A}_{d_i}$, where $\mathbf{d} = (d_1, \ldots, d_n, \ldots)$, which is a continuous function on the space of all sequences $\mathbf{d}$ with the sequence topology. Finite RCP sets of column-stochastic matrices are completely characterized. Some results are given on the problem of algorithmically deciding if a given set $\Sigma$ is an RCP set.

## 1. INTRODUCTION

Consider the two matrices

$$\mathbf{M}_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, \qquad \mathbf{M}_1 = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{1.1}$$

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A sequence in which every element is either $\mathsf{M}_0$ or $\mathsf{M}_1$ is completely characterized by the binary sequence $\mathbf{d} = (d_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ of their indices. For any such sequence, one finds

$$\mathsf{M}_{d_1}\mathsf{M}_{d_2}\cdots\mathsf{M}_{d_n} = \begin{pmatrix} 2^{-n} + \sum_{j=1}^{n} 2^{-j}d_j & \sum_{j=1}^{n} 2^{-j}d_j \\ 1 - 2^{-n} - \sum_{j=1}^{n} 2^{-j}d_j & 1 - \sum_{j=1}^{n} 2^{-j}d_j \end{pmatrix}. \tag{1.2}$$

It follows that these products converge, as $n$ tends to $\infty$, for any sequence $\mathbf{d}$. If we equip the set of binary sequences with the metric

$$D(\mathbf{d}, \mathbf{d}') = \sup\{2^{-n};\, d_n \neq d'_n\},$$

then the infinite products

$$\mathsf{M}(\mathbf{d}) = \lim_{n\to\infty} \mathsf{M}_{d_1}\mathsf{M}_{d_2}\cdots\mathsf{M}_{d_n},$$

depend continuously on $\mathbf{d}$. In fact, if the sequences $\mathbf{d}$ are interpreted as the digits in the binary expansions of numbers $x$ in the unit interval $[0,1]$, which amounts to identifying any two sequences of the type

$$\begin{matrix} d_1 & \cdots & d_n & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots, \\ d_1 & \cdots & d_n & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots, \end{matrix}$$

and one defines $x(\mathbf{d})$ to be the real number in $[0,1]$ associated to $\mathbf{d}$, then $\mathsf{M}(x)$ is well defined for $x \in [0,1]$ and

$$\mathsf{M}(x) = \begin{bmatrix} x & x \\ 1-x & 1-x \end{bmatrix}.$$

This paper studies sets of matrices generalizing the above example. We say that an infinite product $\prod_{i=1}^{\infty}\mathsf{M}_i$ of $n \times n$ matrices *right-converges* if $\lim_{i\to\infty}\mathsf{M}_1\cdots\mathsf{M}_i$ exists, in which case we define

$$\prod_{i=1}^{\infty}\mathsf{M}_i := \lim_{i\to\infty}\mathsf{M}_1\mathsf{M}_2\cdots\mathsf{M}_i. \tag{1.3}$$

A set $\Sigma$ of complex $n \times n$ matrices is said to be an *RCP set* ("right-convergent product") or to have the *RCP property* if all infinite products of members of $\Sigma$ right-converge. If the set $\Sigma$ is finite, $\Sigma = \{\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_{m-1}\}$, then any sequence of elements of $\Sigma$ can be characterized by a sequence $\mathbf{d} = (d_j)_{j \in \mathbb{N}}$ of digits drawn from $\{0, 1, \ldots, m-1\}$. We shall call the set of such sequences $\mathsf{S}_m$. Sequences in $\mathsf{S}_m$ are "close" if they agree on the $N$ consecutive digits $d_1, \ldots, d_N$, where $N$ is "large"; more precisely, we equip $\mathsf{S}_m$ with the metric

$$D(\mathbf{d}, \mathbf{d}') = m^{-r}, \qquad \text{where } r \text{ is the first index such that } d_r \neq d'_r.$$

We shall call the induced topology on $\mathsf{S}_m$ the *sequence topology*.

If the finite set $\Sigma$ has the RCP property, then one can define the *limit function* $\mathsf{M}_\Sigma(\cdot)$ by

$$\mathsf{M}_\Sigma(\mathbf{d}) := \prod_{j=1}^{\infty} \mathsf{A}_{d_j}, \tag{1.4}$$

where $\mathsf{M}_\Sigma(\mathbf{d}) \in M(n, \mathbb{C})$, the space of $n \times n$ complex matrices.

The sequence space $\mathsf{S}_m$ is mapped to $[0,1]$ by viewing $\mathbf{d}$ as an $m$-ary expansion of a real number, i.e., $x : \mathsf{S}_m \to [0,1]$ is given by

$$x(\mathbf{d}) = \sum_{j=1}^{\infty} d_j m^{-j}.$$

This map is continuous, and is one-to-one except at *terminating rationals* $l/m^j$, which have two expansions of the form

$$\begin{matrix} d_1 & \cdots & d_j & 0 & 0 & 0 & \cdots, \\ d_1 & \cdots & d_j - 1 & m-1 & m-1 & m-1 & \cdots. \end{matrix} \tag{1.5}$$

We call an RCP set $\Sigma$ *real-definable* if the images under $\mathsf{M}_\Sigma$ of any two sequences of type (1.5) agree. In this case one obtains a well-defined *real limit function* $\overline{\mathsf{M}}_\Sigma : [0,1] \to \mathsf{M}(n, \mathbb{C})$ given by

$$\overline{\mathsf{M}}_\Sigma(x) := \mathsf{M}_\Sigma(\mathbf{d}(x)), \tag{1.6}$$

where $\mathbf{d}(x)$ is any $m$-ary expansion of $x$.

The RCP set $\Sigma$ is *continuous* if the map $\mathsf{M}_\Sigma$ is continuous with respect to the sequence topology on $\mathsf{S}_m$. It is *real-continuous* if $\Sigma$ is real-definable and the real-limit function $\overline{\mathsf{M}}_\Sigma$ is continuous.

One may define analogous properties for left convergence. A product

$$\prod_{i=1}^{\infty}{}^{L}\mathsf{M}_i$$

of matrices *left-converges* if $\lim_{n\to\infty}\mathsf{M}_i\cdots\mathsf{M}_2\mathsf{M}_1$ exists, in which case we define

$$\prod_{i=1}^{\infty}{}^{L}\mathsf{M}_i = \lim_{n\to\infty}\mathsf{M}_i\cdots\mathsf{M}_2\mathsf{M}_1.$$

A set $\Sigma$ of matrices is an *LCP set* (or has the *LCP property*) if all infinite products of matrices in $\Sigma$ left-converge. If we define the transpose $\Sigma^t$ of $\Sigma$ by

$$\Sigma^t = \{\mathsf{M}^t : \mathsf{M}\in\Sigma\},$$

where $\mathsf{M}^t$ denotes the transpose of $\mathsf{M}$, then it is easy to see that $\Sigma$ is an RCP set if and only if $\Sigma^t$ is an LCP set. For this reason it suffices to study RCP sets. Hartfiel [18] observes that there are RCP sets that are not LCP sets. In particular the matrices $\mathsf{M}_0, \mathsf{M}_1$ in the example (1.1) are an RCP set but not an LCP set.

RCP sets of matrices arise in a surprising number of different contexts. Their limit functions appear in parametrizing various fractal-like objects, for example the continuous, nowhere differentiable snowflake curve of von Koch [37], and curves constructed by de Rham [30–32], as we show in Section 6. More generally, attractors of hyperbolic iterated function systems on $\mathbb{R}^n$ given by affine mappings can be similarly parametrized; see Barnsley [2]. (The parametrization uses the addresses of points discussed in [2, §§4.1, 4.2].) RCP sets arise in computer-aided geometric design in constructing parametrized curves and surfaces by subdivision or refinement algorithms [3], as was first observed by Micchelli and Prautzsch [21–23]. They occur also in the study of nonhomogeneous Markov chains [16, 35] and in probabilistic automata [26]. Our interest in RCP sets of matrices arose from study of lattice two-scale difference equations [8, 9]. These are functional equations of the type

$$f(x) = \sum_{n=N_1}^{N_2} c_n f(kx - n), \tag{1.7}$$

where $k$ is an integer strictly large than 1, and where $N_1 \leqslant N_2$ are both finite. In [9] we showed that $L^1$-solutions of such equations can often be constructed using the limit function of an RCP-set $\Sigma$ associated to the equation (1.7), consisting of $k$ matrices of size $N = [(N_2 - N_1)/(k-1)]$. The convergence properties of the associated infinite products permit one to analyze the properties of such solutions $f(x)$; see [9]. Lattice two-scale difference equations with $L^1$-solutions arise in interpolation schemes studied by Deslauriers and Dubuc [11–13]; in the construction of orthonormal bases of compactly supported wavelets [7], which are useful in signal analysis and numerical analysis; and in the construction of various splines, e.g., the normalized $B$-spline of degree $n$ satisfies a two-scale difference equation.

The large variety of such examples motivates the basic object of this paper, which is to characterize RCP sets. In particular, we completely characterize RCP sets having a continuous limit function; we prove in this case that all infinite products converge uniformly at a geometric rate depending on $\Sigma$.

The contents of the paper are as follows. Sections 2 and 3 derive necessary conditions for infinite or finite sets $\Sigma$ to have the RCP property. These are conditions on the eigenvalues and left eigenspaces of matrices in $\Sigma$ and finite products of these matrices. Section 3 gives a necessary condition for a finite set $\Sigma$ to be an RCP set, which is that the joint spectral radius $\hat{\rho}(\Sigma)$ of $\Sigma$ satisfies $\hat{\rho}(\Sigma) \leqslant 1$. The concept of joint spectral radius of a set of matrices was introduced by Rota and Strang [34], in a more general setting.

Section 4 derives necessary and sufficient conditions for a finite RCP set to have a continuous limit function. These conditions split into two parts: the matrices $A_j$ in $\Sigma$ must all have the same left eigenspace $E_1$ for the eigenvalue 1, and when restricted to a complement of $E_1$, the set of matrices has the property that it *contracts*, in the sense that its joint spectral radius is strictly smaller than 1. Because of this contraction property, infinite products of elements of a finite continuous RCP set converge at a geometric rate. The example at the start of this introduction illustrates these two properties: the left 1-eigenspace, for both $\mathsf{M}_0$ and $\mathsf{M}_1$, consists of the multiples of $(1,1)$, while the restrictions of $\mathsf{M}_0$ and $\mathsf{M}_1$ to any complement of this eigenspace are equivalent to multiplication by $\frac{1}{2}$ (these complements are 1-dimensional), so that the joint spectral radius of these restrictions is also $\frac{1}{2}$. The geometric rate of the convergence is clear from (1.2). We also derive extra necessary and sufficient conditions ensuring that the limit function is real-continuous.

Section 5 treats RCP sets whose limit function need not be continuous. A set of matrices is *product-bounded* if there exists a uniform bound for all the finite products of elements of the set. We give necessary and sufficient conditions for a finite set $\Sigma$ to be a product-bounded RCP set.

Section 6 describes in detail various examples of RCP sets. In particular Theorem 6.1 gives necessary and sufficient conditions for a finite set of column-stochastic matrices to be an RCP set having a limit function that consists of matrices of rank at most one.

Section 7 describes similarities and differences between the results of this paper and parallel results in the theory of random matrices, a subject on which there is a large literature [6].

The methods of proof of this paper for RCP sets having a real-continuous limit function are analogous to those used in recent work of Micchelli and Prautzch [23]. They study vector-valued functions

$$\boldsymbol{\psi}(x) = \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{bmatrix}$$

on $[0,1]$ satisfying the functional equations

$$\boldsymbol{\psi}(x) = B_i \boldsymbol{\psi}(px - i), \qquad x \in \left[\frac{i}{p}, \frac{i+1}{p}\right],$$

where $\Sigma = \{B_0, \dots, B_{p-1}\}$ is a set of $n \times n$ matrices, and derive necessary and sufficient conditions on $\Sigma$ for a solution $\boldsymbol{\psi}(x)$ to exist and be in $C^k([0,1])$ for $k = 0, 1, 2, \dots$. The conditions for $\boldsymbol{\psi}(x)$ to be in $C^0([0,1])$ (Theorem 5.1 of [23]) resemble those of Theorem 4.3. In particular, using Theorem 4.2 of this paper, their result implies that a necessary condition for a $C^0([0,1])$ solution to exist is that the set $\Sigma' = \{B_0 P, \dots, B_{p-1} P\}$ be an RCP set, where $P$ is the projection onto a certain subspace $\mathsf{S}$ of $\mathbb{R}^n$ which they define.

## 2. RCP SETS: NECESSARY CONDITIONS ON EIGENVALUES AND EIGENSPACES

Our first object is to derive necessary conditions for a (finite or infinite) set $\Sigma$ to be an RCP set, which involve restrictions on the eigenvalues and eigenspaces of finite products drawn from $\Sigma$.

We first observe that similarity transformations take RCP-sets to RCP-sets. Given two matrices $\mathsf{X}, \mathsf{Y}$, define

$$\mathsf{X}\Sigma\mathsf{Y} := \{\mathsf{XAY} : \mathsf{A} \in \Sigma\}.$$

In particular $\mathsf{S}\Sigma\mathsf{S}^{-1}$ denotes a similarity transformation applied to $\Sigma$.

LEMMA 2.1. *If* $\mathsf{S}$ *is an invertible matrix and* $\Sigma$ *is an RCP set, then* $\mathsf{S}\Sigma\mathsf{S}^{-1}$ *is an RCP set. If in addition* $\Sigma$ *is finite, then its limit function satisfies*

$$\mathsf{M}_{\mathsf{S}\Sigma\mathsf{S}^{-1}}(\mathbf{d}) = \mathsf{S}\mathsf{M}_{\Sigma}(\mathbf{d})\mathsf{S}^{-1}, \qquad \mathbf{d} \in \mathsf{S}_m.$$

*Proof.* Immediate. ■

It is natural to consider necessary conditions involving eigenvalues because they are similarity invariants.

THEOREM 2.1. *Let* $\Sigma$ *be any RCP set. Then*:

(1) *The eigenvalues* $\lambda$ *of any finite product of matrices in* $\Sigma$ *satisfy* $|\lambda| \leqslant 1$. *If* $|\lambda| = 1$ *then* $\lambda = 1$.

(2) *The left* 1*-eigenspace* $E_1(\mathsf{M})$ *of each matrix* $\mathsf{M}$ *in* $\Sigma$ *is simple, i.e. has a basis of eigenvectors.*

(3) *The left* 1*-eigenspace* $E_1(\mathsf{B})$ *of any finite product* $\mathsf{B} = \prod_{i=1}^{k}\mathsf{M}_i$ *is* $\bigcap_{i=1}^{k} E_1(\mathsf{M}_i)$.

*Proof.* (1): If any finite product $\mathsf{B}$ has some $|\lambda| > 1$ or $|\lambda| = 1$ and $\lambda \neq 1$, then the periodic infinite product with period $\mathsf{B} = \prod_{i=1}^{n}\mathsf{M}_i$ does not converge. Indeed, if $\mathbf{v}$ is a left eigenvector with eigenvalue $\lambda$, then

$$\mathbf{v}\mathsf{B}^j = \lambda^j \mathbf{v},$$

so that $\lim_{j\to\infty}\mathbf{v}\mathsf{B}^j$ does not exist, in both cases.

(2): Given $\mathsf{M}$ in $\Sigma$, suppose that $E_1(\mathsf{M})$ has no basis of eigenvectors. Consequently there exist vectors $\mathbf{v}_1, \mathbf{v}_2 \in E_1(\mathsf{M})$ such that

$$\mathbf{v}_1\mathsf{M} = \mathbf{v}_1,$$

$$\mathbf{v}_2\mathsf{M} = \mathbf{v}_1 + \mathbf{v}_2.$$

Then $\mathsf{v}_2\mathsf{M}^j = j\mathsf{v}_1 + \mathsf{v}_2$, so that $\lim_{j\to\infty}\mathsf{v}_2\mathsf{M}^j$ does not exist, which contradicts $\Sigma$ being an RCP set.

(3): Given $\mathsf{B} = \prod_{i=1}^{k}\mathsf{M}_i$, it is immediate that $\bigcap_{i=1}^{k} E_1(\mathsf{M}_i) \subseteq E_1(\mathsf{B})$. To prove the opposite inclusion, consider the periodic infinite product with period $\mathsf{B}$, which is $\mathsf{M}^{(\infty)} = \lim_{j\to\infty}\mathsf{M}^{(j)}$, where

$$\mathsf{M}^{(j)} = \prod_{i=1}^{j} \mathsf{M}_{i-n[(i-1)/n]}.$$

We claim that

$$\mathsf{M}^{(\infty)}\mathsf{M}_i = \mathsf{M}^{(\infty)}, \qquad 1 \leqslant i \leqslant n. \tag{2.1}$$

Indeed,

$$M^{(\infty)} = \lim_{j\to\infty}\mathsf{M}^{(jk+i)} = \lim_{j\to\infty}\mathsf{M}^{(jk+i-1)}\mathsf{M}_i = \mathsf{M}^{(\infty)}\mathsf{M}_i.$$

Now suppose that $\mathsf{v} \in E_1(\mathsf{B})$. Then

$$\mathsf{v}\mathsf{M}^{(\infty)} = \lim_{j\to\infty}\mathsf{v}\mathsf{M}^{(jk)} = \lim_{j\to\infty}\mathsf{v}\mathsf{B}^j = \mathsf{v}.$$

Then (2.1) gives

$$\mathsf{v}\mathsf{M}_i = (\mathsf{v}\mathsf{M}^{(\infty)})\mathsf{M}_i = \mathsf{v}\left(\mathsf{M}^{(\infty)}\mathsf{M}_i\right) = \mathsf{v}\mathsf{M}^{(\infty)} = \mathsf{v}.$$

Hence $\mathsf{v} \in \bigcap_{i=1}^{k} E_1(\mathsf{M}_i)$. ∎

Now suppose that $\Sigma$ is a finite RCP set. For any finite subset $\Sigma'$ of $\Sigma$ let

$$E_1(\Sigma') = \bigcap_{\mathsf{M}\in\Sigma'} E_1(\mathsf{M}).$$

be the common 1-eigenspace of all elements of $\Sigma'$. There are some constraints on the behavior of the limit function.

THEOREM 2.2. *Let* $\Sigma = \{\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_{m-1}\}$ *be a finite RCP set. For any sequence* $\mathbf{d} = \{d_1, d_2, d_3, \ldots\}$ *of digits* $d_i \in \{0, 1, \ldots, m-1\}$ *define*

$$\mathrm{Unb}(\mathbf{d}) = \{d : d = d_i \textit{ for infinitely many } i\},$$

*and* $\Sigma' = \Sigma'(\mathbf{d}) = \{\mathsf{A}_d : d \in \mathrm{Unb}(\mathbf{d})\}$. *Then* $\mathsf{M}_\Sigma(\mathbf{d}) = \prod_{i=1}^{\infty} \mathsf{A}_{d_i}$ *satisfies*

$$\mathsf{M}_\Sigma(\mathbf{d})\mathsf{A} = \mathsf{M}_\Sigma(\mathbf{d}) \qquad \text{if} \quad \mathsf{A} \in \Sigma'(\mathbf{d}), \tag{2.2}$$

*so that each row of* $\mathsf{M}_\Sigma(\mathbf{d})$ *belongs to* $E_1(\Sigma')$. *If in addition all digits of* $\mathbf{d}$ *are in* $\mathrm{Unb}(\mathbf{d})$, *then*

$$\mathbf{v}\mathsf{M}_\Sigma(\mathbf{d}) = \mathbf{v} \qquad \text{for all} \quad \mathbf{v} \in E_1(\Sigma'(\mathbf{d})). \tag{2.3}$$

*Proof*. Let $\mathsf{A}^{(k)} = \prod_{i=1}^{k} \mathsf{A}_{d_i}$. Suppose that $d \in \mathrm{Unb}(\mathbf{d})$ and that $d_j = d$ for $j$ in an infinite set $J$. Then

$$\mathsf{M}_\Sigma(\mathbf{d}) = \lim_{\substack{j \to \infty \\ j \in J}} \mathsf{A}^{(j)} = \lim_{\substack{j \to \infty \\ j \in J}} \mathsf{A}^{(j-1)}\mathsf{A}_d = \mathsf{M}_\Sigma(\mathbf{d})\mathsf{A}_d.$$

This proves (2.2). The inclusion $E_1(\Sigma') \subseteq E_1(\mathsf{M}_\Sigma(\mathbf{d}))$ given by (2.3) is obvious. ■

## 3. RCP SETS: NECESSARY CONDITIONS USING MATRIX NORMS

We derive further necessary conditions for a finite set $\Sigma$ to be an RCP set which involve extensions of the concept of spectral radius to a set of matrices. The *spectral radius* $\rho(\mathsf{M})$ of a square matrix $\mathsf{M}$ is

$$\rho(\mathsf{M}) := \max\{|\lambda| : \lambda \text{ an eigenvalue of } \mathsf{M}\}. \tag{3.1}$$

There are two natural generalizations of this concept to a finite set of matrices $\Sigma$: the generalized spectral radius $\rho(\Sigma)$ and the joint spectral radius $\hat{\rho}(\Sigma)$.

The *generalized spectral radius* $\rho(\Sigma)$ of any set of matrices $\Sigma$ is

$$\rho(\Sigma) := \limsup_{k \to \infty} \left(\rho_k(\Sigma)\right)^{1/k}, \tag{3.2}$$

where

$$\rho_k(\Sigma) := \sup\left\{\rho\left(\prod_{i=1}^{k} \mathsf{M}_i\right) : \mathsf{M}_i \in \Sigma \text{ for } 1 \leqslant i \leqslant k\right\}. \tag{3.3}$$

The second generalization uses matrix norms. A *matrix norm* $\|\cdot\|$ is a mapping $\|\cdot\|: \mathsf{M}(n,\mathbb{C}) \to \mathbb{R}_{\geqslant 0}$ such that

(1) $\|\mathsf{M}_1 + \mathsf{M}_2\| \leqslant \|\mathsf{M}_1\| + \|\mathsf{M}_2\|$,
(2) $\|\mathsf{M}_1\mathsf{M}_2\| \leqslant \|\mathsf{M}_1\| \, \|\mathsf{M}_2\|$,
(3) $\|\lambda\mathsf{M}\| = |\lambda| \, \|\mathsf{M}\|$ for all $\lambda \in \mathbb{C}$,
(4) $\|\mathsf{M}\| = 0$ implies that $\mathsf{M}$ is a zero matrix.

It is well known (see [20]) that any two matrix norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in the sense that there are constants $0 < c_1 < c_2$ such that

$$c_1\|\mathsf{M}\|_1 < \|\mathsf{M}\|_2 < c_2\|\mathsf{M}\|_1 \tag{3.4}$$

holds for all $\mathsf{M}$.

A particularly useful matrix norm is the *spectral norm* defined by

$$\|\mathsf{M}\|_s := \sup_{\|\mathbf{x}\| = 1} \|\mathsf{M}\mathbf{x}\|, \tag{3.5}$$

where $\|\mathbf{x}\|$ is the Euclidean norm on $\mathbb{R}^n$. The spectral norm has the additional property that it is defined for matrices of all sizes, including nonsquare matrices, and properties (1)–(4) above hold in all cases where they make sense, e.g., (2) holds when $\mathsf{M}_1$ is $m \times r$ and $\mathsf{M}_2$ is $r \times n$. Furthermore, for any block-partitioned matrix

$$\mathsf{M} = \begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{21} & \mathsf{M}_{22} \end{bmatrix},$$

one has

$$\|\mathsf{M}\|_s \leqslant \|\mathsf{M}_{11}\|_s + \|\mathsf{M}_{12}\|_s + \|\mathsf{M}_{21}\|_s + \|\mathsf{M}_{22}\|_s,$$

and for $i, j \in \{1,2\}$,

$$\|\mathsf{M}_{ij}\|_s \leqslant \|\mathsf{M}\|_s.$$

Any matrix norm gives an upper bound for the spectral radius, for one has [20, p. 359]

$$\rho(\mathsf{M}) \leqslant \|\mathsf{M}\|. \tag{3.6}$$

Furthermore, it is well known that (see [34])

$$\rho(\mathsf{M}) = \lim_{k \to \infty} \|\mathsf{M}^k\|^{1/k}. \tag{3.7}$$

The *joint spectral radius* $\hat{\rho}(\Sigma)$ is defined by

$$\hat{\rho}(\Sigma) := \limsup_{k \to \infty} \left(\hat{\rho}_k(\Sigma, \|\cdot\|)\right)^{1/k}, \tag{3.8}$$

where $\|\cdot\|$ is any matrix norm and

$$\hat{\rho}_k(\Sigma, \|\cdot\|) := \sup\left\{\left\|\prod_{i=1}^{k} \mathsf{M}_i\right\| : \mathsf{M}_i \in \Sigma \text{ for } 1 \leqslant i \leqslant k\right\}. \tag{3.9}$$

The definition of $\hat{\sigma}(\Sigma)$ is independent of the norm used in (3.9), as is easily proved using (3.4). The notion of *joint spectral radius* is a special case of the concept of joint spectral radius of a bounded subset of a normed algebra introduced by Rota and Strang [34].

Both the generalized spectral radius and joint spectral radius coincide with the usual spectral radius when $\Sigma$ consists of a single matrix $\mathsf{M}$.

We derive basic inequalities relating the quantities $\rho_k(\Sigma)$, $\rho(\Sigma)$, $\hat{\rho}_k(\Sigma, \|\cdot\|)$, and $\hat{\rho}(\Sigma)$. First, observe that similarity transformations leave $\rho_k$ and $\rho$ unchanged. However, $\hat{\rho}_k$ may change, and one has the bounds

$$\left(\|\mathsf{S}\|_s \|\mathsf{S}^{-1}\|_s\right)^{-1} \hat{\rho}_k(\Sigma) \leqslant \hat{\rho}_k(\mathsf{S}\Sigma\mathsf{S}^{-1}) \leqslant \hat{\rho}_k(\Sigma)\left(\|\mathsf{S}\|_s \|\mathsf{S}^{-1}\|_s\right). \tag{3.10}$$

As a consequence (3.8) gives

$$\hat{\rho}(\mathsf{S}\Sigma\mathsf{S}^{-1}) = \hat{\rho}(\Sigma), \tag{3.11}$$

so $\hat{\rho}$ is invariant under similarity transformations.

LEMMA 3.1. *For any set of matrices $\Sigma$, any $k \geqslant 1$, and any matrix norm $\|\cdot\|$,*

$$\rho_k(\Sigma)^{1/k} \leqslant \rho(\Sigma) \leqslant \hat{\rho}(\Sigma) \leqslant \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}. \tag{3.12}$$

*Proof.* The inequality $\rho(\Sigma) \leqslant \hat{\rho}(\Sigma)$ follows directly from (3.6), on comparing the definitions term by term.

To prove the leftmost inequality in (3.12), observe for any $m \geqslant 1$ that

$$\left(\rho_k(\Sigma)\right)^m \leqslant \rho_{mk}(\Sigma),$$

for all matrices in $\rho_k(\Sigma)$ raised to the $m$th power appear in the definition (3.3) of $\rho_{mk}(\Sigma)$, and $\rho(\mathsf{M}^m) = \rho(\mathsf{M})^m$. Hence $\rho_k(\Sigma)^{1/k} \leqslant \rho_{mk}(\Sigma)^{1/mk}$, so letting $m \to \infty$ gives the result.

To prove the rightmost inequality in (3.12), we may assume that there is a finite bound $b_0$ such that $\|\mathsf{M}\| \leqslant b_0$ for all $\mathsf{M}$ in $\Sigma$, for if not then $\hat{\rho}_k(\mathsf{M}, \|\cdot\|) = \infty$ and the inequality is vacuous. Thus for any $l \geqslant k$, write $l = mk + j$ with $0 \leqslant j \leqslant k - 1$, and one has

$$\|\mathsf{M}_1 \cdots \mathsf{M}_l\| \leqslant \prod_{i=0}^{m-1} \|\mathsf{M}_{ik+1}\mathsf{M}_{ik+2} \cdots \mathsf{M}_{(i+1)k}\| \prod_{i=1}^{j} \|\mathsf{M}_{mk+i}\|$$

$$\leqslant b_0^j\left(\hat{\rho}_k(\Sigma, \|\cdot\|)\right)^m.$$

Taking the supremum over all products of length $l$, one easily obtains

$$\hat{\rho}_l(\Sigma, \|\cdot\|)^{1/l} \leqslant b_0^{j/l}\hat{\rho}_k(\Sigma, \|\cdot\|)^{m/l} = \left(b_0\hat{\rho}_k(\Sigma, \|\cdot\|)^{-1/k}\right)^{j/l}\hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}.$$

Letting $l \to \infty$ gives $(b_0\hat{\rho}_k(\Sigma, \|\cdot\|)^{-1/k})^{j/l} \to 1$, so that

$$\hat{\rho}(\Sigma) = \limsup_{l \to \infty} \hat{\rho}_l(\Sigma, \|\cdot\|)^{1/l} \leqslant \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}. \qquad \blacksquare$$

Theorem 2.1 shows that $\rho(\Sigma) \leqslant 1$ for any RCP set. The main result of this section is a strengthened bound for finite RCP sets.

THEOREM 3.1. *If $\Sigma$ is a finite RCP set, then*

$$\hat{\rho}(\Sigma) \leqslant 1. \tag{3.13}$$

*Proof.* We claim that there exists an infinite sequence $\mathbf{d} = (d_1, d_2, \dots)$ of digits such that for

$$\mathsf{A}^{(k)} = \prod_{i=1}^{k} \mathsf{A}_{d_i}$$

one has

$$\|\mathsf{A}^{(k)}\|^{1/k} \geqslant \hat{\rho}(\Sigma), \qquad k = 1,2,3,\dots. \tag{3.14}$$

To prove this, we make a directed graph which is a rooted tree, whose vertices correspond to certain finite products of the $\mathsf{A}_i$. The root is the empty set $\varnothing$, and there is a directed edge to $\mathsf{A}_d$ if $\|\mathsf{A}_d\| \geqslant \hat{\rho}(\Sigma)$, and from $\mathsf{B}_k = \mathsf{A}_{i_1} \cdots \mathsf{A}_{i_k}$ to $\mathsf{B}_k \mathsf{A}_{i_{k+1}}$ if $\|\mathsf{A}_{i_1} \cdots \mathsf{A}_{i_j}\|^{1/j} \geqslant \hat{\rho}(\Sigma)$ for $1 \leqslant j \leqslant k+1$. Let $\mathscr{T}$ denote the set of vertices in this tree. We show $\mathscr{T}$ is infinite. For suppose not, and define the finite set

$$\mathscr{C} = \{\mathsf{B}\mathsf{A}_i : \mathsf{B} \in \mathscr{T}, \mathsf{B}\mathsf{A}_i \notin \mathscr{T}\}.$$

$\mathscr{C}$ is a *prefix code*, i.e., any infinite string $(\mathsf{A}_{d_1}, \dots, \mathsf{A}_{d_n}, \dots)$ has a unique finite prefix that is one of the words in $\mathscr{C}$. By definition of $\mathscr{T}$ any word $\mathsf{W} = \mathsf{A}_{i}, \dots, \mathsf{A}_{i_k}$ in $\mathscr{C}$ has $\|\mathsf{W}\|^{1/k} < \hat{\rho}(\Sigma)$. Let $\hat{\rho}(\Sigma) - \alpha$ represent the maximal value of such $\|\mathsf{W}\|^{1/k}$, and $l$ the maximal length of a word in $\mathscr{C}$. Now any finite product $\mathsf{A}_{d_1} \cdots \mathsf{A}_{d_j}$ of length $j$ factors into at least $[j/l]$ words in $\mathscr{C}$ times a remainder word of length $< l$. Hence

$$\begin{aligned} \rho_j(\Sigma) &\leqslant \max\left\{\|\mathsf{A}_{d_1} \cdots \mathsf{A}_{d_j}\| : 0 \leqslant d_i \leqslant m-1,\ 1 \leqslant i \leqslant j\right\} \\ &\leqslant (\hat{\rho}(\Sigma) - \alpha)^{j-l} \left(\max(1, \|\mathsf{A}_0\|, \dots, \|\mathsf{A}_{m-1}\|)\right)^{l}. \end{aligned}$$

Hence $\limsup_{j \to \infty} \rho_j(\Sigma)^{1/j} \leqslant \hat{\rho}(\Sigma) - \alpha$, contradicting the definition of $\hat{\rho}(\Sigma)$. So $\mathscr{T}$ is infinite.

Now the tree $\mathscr{T}$ has finite branching at each vertex (at most $m$ edges). Hence by König's infinity lemma (see [24]) $\mathscr{T}$ has an infinite chain, which is $\mathbf{d} = (d_1, d_2, \dots)$, for which (3.14) holds, proving the claim.

Since the infinite product $\mathsf{A}^{(\infty)} = \prod_{i=1}^{\infty} \mathsf{A}_{d_i}$ converges, there is a finite bound $\Delta$ with $\|\mathsf{A}^{(k)}\| \leqslant \Delta$ for all $k \geqslant 1$. Hence by (3.14),

$$\hat{\rho}(\Sigma) \leqslant \limsup_{k \to \infty} \Delta^{1/k} = 1. \qquad \blacksquare$$

It seems possible that the concepts of generalized spectral radius and joint spectral radius coincide for finite sets $\Sigma$.

GENERALIZED SPECTRAL-RADIUS CONJECTURE. For all finite sets $\Sigma$ one has

$$\rho(\Sigma) = \hat{\rho}(\Sigma).$$

This conjecture does not hold for infinite sets, e.g.,

$$\Sigma = \left\{ \begin{pmatrix} \frac{1}{2} & 2^n \\ 0 & \frac{1}{2} \end{pmatrix} : n \geqslant 1 \right\}$$

has $\rho(\Sigma) = \frac{1}{2}$, while $\hat{\rho}(\Sigma) = \hat{\rho}_1(\Sigma) = +\infty$.

## 4. RCP SETS HAVING A CONTINUOUS LIMIT FUNCTION

This section characterizes finite RCP sets having continuous or real-continuous limit functions. We first study finite RCP sets $\Sigma$ whose limit function is identically zero.

THEOREM 4.1. *Let*

$$\Sigma = \{\mathsf{A}_0, \dots, \mathsf{A}_{m-1}\} \tag{4.1}$$

*be a finite RCP set. The following conditions are equivalent*:

(1) $\Sigma$ *is an RCP set whose real limit function* $\overline{\mathsf{M}}_\Sigma$ *is identically zero on* $[0,1]$.

(2) $\Sigma$ *is an RCP set whose limit function* $\mathsf{M}_\Sigma$ *is identically zero on* $\mathsf{S}_m$.

(3) *The joint spectral radius* $\hat{\rho}(\Sigma)$ *satisfies*

$$\hat{\rho}(\Sigma) < 1. \tag{4.2}$$

*Proof.* (2) ⇒ (1) is trivial.

(3) ⇒ (2): Since $\hat{\rho}(\Sigma) < 1$, one has $\hat{\rho}_t(\Sigma) < 1$ for some finite $t$. For $j = lt + k$ with $0 \leqslant k < t$,

$$\|\mathsf{A}_{d_1} \cdots \mathsf{A}_{d_j}\|_s \leqslant \hat{\rho}_t(\Sigma)^l a^k \leqslant \max[1, a^t]\hat{\sigma}_t(\Sigma)^l, \tag{4.3}$$

where $a = \hat{\rho}_1(\Sigma) = \max(\|\mathsf{A}_i\|_s)$, so all infinite products converge to the zero matrix at a geometric rate as $l \to \infty$.

(1) ⇒ (3): Suppose (3) is false, so that $\hat{\rho}(\Sigma) \geqslant 1$. Then $\hat{\rho}(\Sigma) = 1$ by Theorem 3.1, and the proof of Theorem 3.1 shows that there exists $\mathsf{A}^{(\infty)} = \prod_{i=1}^{\infty} \mathsf{A}_{d_i}$ such that $\mathsf{A}^{(k)} = \prod_{i=1}^{k} \mathsf{A}_{d_i}$ satisfies

$$\|\mathsf{A}^{(k)}\| \geqslant \hat{\rho}(\Sigma) = 1, \qquad \text{all} \quad k \geqslant 1.$$

Then

$$\|\mathsf{A}^{(\infty)}\| = \lim_{k \to \infty} \|\mathsf{A}^{(k)}\| \geqslant 1. \tag{4.4}$$

If $\alpha = \sum_{i=1}^{\infty} d_i m^{-i}$ is a nonterminating real number, then by hypothesis (1) $\mathsf{A}^{(\infty)} = \overline{\mathsf{M}}_{\Sigma}(\alpha) = \mathbf{0}$, contradicting (4.4). If $\alpha$ is terminating, then necessarily all $\mathsf{A}_{d_i}$ are constant from some point on, equal to either $\mathsf{A}_0$ or $\mathsf{A}_{m-1}$. Suppose it is $\mathsf{A}_0$. Since $\mathsf{M}_{\Sigma}(0) = \mathbf{0}$, we have $(\mathsf{A}_0)^k \to \mathbf{0}$ as $k \to \infty$. Hence $\lim_{k \to \infty} \|\mathsf{A}^{(k)}\| = 0$, contradicting (4.3). If all $\mathsf{A}_{d_i} = \mathsf{A}_{m-1}$ from some point on, we use $\mathsf{M}_{\Sigma}(1) = \mathbf{0}$ to contradict (4.4) in the same way. Thus (4.4) is contradicted in all cases, so (3) is proved. ■

Corollary 4.1a. *If $\Sigma$ is a finite RCP set whose limit function is identically zero, then all infinite products from $\Sigma$ uniformly converge to $\mathbf{0}$ at a geometric rate.*

*Proof.* This follows from (4.3). ■

Next we treat finite RCP sets having a continuous limit function.

Theorem 4.2. *Let $\Sigma$ be a finite set of $n \times n$ matrices. The following conditions are equivalent:*

(1) *$\Sigma$ is an RCP set whose limit function $\mathsf{M}_{\Sigma}$ is continuous.*

(2) *All matrices $\mathsf{A}_i$ in $\Sigma$ have the same 1-eigenspace $E_1 = E_1(\mathsf{A}_i)$, and this eigenspace is simple for all $\mathsf{A}_i$. There exists a vector space $V$ with*

$\mathbb{R}^n = E_1 + V$, *having the property that if* $\mathsf{P}_V$ *is the orthogonal projection onto* $V$ *then* $\mathsf{P}_V\Sigma\mathsf{P}_V$ *is an RCP set whose limit function is identically zero.*

(3) *The same as* (2), *except* $\mathsf{P}_V\Sigma\mathsf{P}_V$ *is an RCP set with limit function identically zero for all vector spaces* $V$ *with* $\mathbb{R}^n = E_1 + V$ *and* $\dim(V) = n - \dim(E_1)$.

*Proof.* (3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): One must have $\dim(V) = n - d$, where $d = \dim(E_1)$. For certainly $\dim(V) \geqslant n - d$, and if strict inequality occurs then $V \cap E_1$ contains a nonzero vector $\mathbf{w}$, which is then in $E_1(\mathsf{P}_V\Sigma\mathsf{P}_V)$, whence $\mathbf{w}\mathsf{M}_{\mathsf{P}_V\Sigma\mathsf{P}_V}(\mathbf{d}) = \mathbf{w}$ for all $\mathbf{d}$, contradicting $\mathsf{M}_{\mathsf{P}_V\Sigma\mathsf{P}_V}(\mathbf{d})$ being identically zero.

By Lemma 2.1, if hypothesis (2) applies to $\Sigma$ with vector space $V$, then it applies to $\mathsf{S}^{-1}\Sigma\mathsf{S}$ with vector space $\mathsf{S}^{-1}V\mathsf{S}$. Similarly, the desired conclusion (1) also is preserved by similarity transformation. Thus, without loss of generality, by making a suitable similarity transformation we may reduce to the case where $\{\mathbf{e}_i : 1 \leqslant i \leqslant n - d\}$ is a basis of $V$, where $\mathbf{e}_i$ is the $i$th row of the identity matrix. In this case hypothesis (2) implies that $\mathsf{A}_i$ has the block form

$$\mathsf{A}_i = \begin{bmatrix} \mathsf{I}_d & \mathbf{0} \\ \mathsf{C}_i & \tilde{\mathsf{A}}_i \end{bmatrix}, \qquad 0 \leqslant i \leqslant m-1, \tag{4.5}$$

where $\mathsf{I}_d$ is a $d \times d$ identity matrix, $\mathbf{0}$ is an $d \times (n-d)$ zero matrix, and $\tilde{\Sigma} = \{\tilde{\mathsf{A}}_i : 0 \leqslant i \leqslant m-1\}$ is an RCP set of $(n-d)\times(n-d)$ matrices whose limit function is identically zero.

Now let $\mathbf{d} = (d_1, d_2, \ldots) \in \mathsf{S}_m$ and set $\mathsf{M}^{(k)} = \prod_{i=1}^{k} \mathsf{A}_{d_i}$. Then $\mathsf{M}^{(k)}$ has the block form

$$\mathsf{M}^{(k)} = \begin{bmatrix} \mathsf{I}_d & \mathbf{0} \\ \mathsf{C}^{(k)} & \tilde{\mathsf{M}}^{(k)} \end{bmatrix},$$

where

$$\begin{aligned} \mathsf{C}^{(k)} &= \mathsf{C}_{d_1} + \sum_{i=2}^{k} \tilde{\mathsf{M}}^{(i-1)}\mathsf{C}_{d_i}, \\ \tilde{\mathsf{M}}^{(k)} &= \prod_{i=1}^{k} \tilde{\mathsf{A}}_{d_i}. \end{aligned} \tag{4.6}$$

By Corollary 4.1a, $\|\tilde{\mathsf{M}}^{(k)}\|$ uniformly converges to zero geometrically in $k$.

This implies that $C^{(k)}$ converges to a limit $C^{(\infty)}$ and [using (4.3) and (4.6)] that

$$\|C^{(k)} - C^{(\infty)}\|_s \leqslant \sum_{i=k+1}^{\infty} \|\tilde{M}^{(i-1)} C_{d_i}\|_s$$

$$\leqslant \beta\left(1 - \hat{\rho}_t(\tilde{\Sigma})\right)^{-1} \hat{\rho}_t(\tilde{\Sigma})^k, \tag{4.7}$$

where $\beta = \max(\|C_i\|_s) \leqslant \hat{\rho}_1(\Sigma)$ and $t$ is chosen so that $\hat{\rho}_t(\tilde{\Sigma}) < 1$. Thus the infinite product converges uniformly to

$$M^{(\infty)} = \begin{bmatrix} I & 0 \\ C^{(\infty)} & 0 \end{bmatrix}, \tag{4.8}$$

and the limit function $M_\Sigma$ is continuous by (4.7).

(1) ⇒ (3): By Theorem 2.1 each matrix $A_i$ has a simple left eigenspace $E_1(A_i)$. We claim that the condition that $M_\Sigma$ is continuous implies that each $E_1(A_i)$ is equal to $E_1(\Sigma) = \bigcap_{i=0}^{m-1} E_1(A_i)$. Suppose not, and by renumbering the $A_i$ if necessary let $E_1(\Sigma) \subsetneqq E_1(A_0)$. For $\mathbf{d}^0 = (0,0,0,\dots) \in S_m$ one has

$$M_\Sigma(\mathbf{d}^0) = A_0^{(\infty)}.$$

By Theorem 2.2 one has $\mathbf{v}M_\Sigma(\mathbf{d}^0) = \mathbf{v}$ for all $\mathbf{v} \in E_1(A_0)$; hence

$$\operatorname{rank}\left(M_\Sigma(\mathbf{d}^0)\right) \geqslant \dim\left(E_1(A_0)\right).$$

Now Theorem 2.2 also shows that each row of $M_\Sigma(\mathbf{d}^0)$ is in $E_1(A_0)$, so that $\operatorname{rank}(M_\Sigma(\mathbf{d}^0)) = \dim(E_1(A_0))$ and the rows of $M_\Sigma(\mathbf{d}^0)$ span $E_1(A_0)$. Consequently some row $\mathbf{w}$ of $M_\Sigma(\mathbf{d}^0)$ is not in $E_1(\Sigma)$. Now let $\mathbf{d}^k$ be the sequence $0^k(0\,1\,2 \cdots (m-1))^\infty$, whose first $k$ entries are zero, and whose remaining entries cycle periodically through all $m$ digits. Then the first part of Theorem 2.2 gives

$$M_\Sigma(\mathbf{d}^k)A_i = M_\Sigma(\mathbf{d}^k), \qquad 0 \leqslant i \leqslant m-1,$$

whence all rows of $M_\Sigma(\mathbf{d}^k)$ are in $E_1(\Sigma)$. Now $\mathbf{d}^k \to \mathbf{d}^0$ as $k \to \infty$, but if $\lim_{k\to\infty} M_\Sigma(\mathbf{d}^k)$ exists, all its rows must be in $E_1(\Sigma)$, and hence in all cases

$$\lim_{k\to\infty} M_\Sigma(\mathbf{d}^k) \neq M_\Sigma(\mathbf{d}^0),$$

and the limit function is not continuous on $\mathsf{S}_m$, which is a contradiction that proves the claim.

Now let $V$ be given with $\mathbb{R}^n = E_1 + V$ and $\dim(V) = n - d$ with $d = \dim(E_1)$. Using a suitable similarity transformation [which preserves hypotheses (1) and (3)], we may without loss of generality reduce to the case where $\{\mathbf{e}_i : 1 \leqslant i \leqslant d\}$ is a basis of $E_1$, $\{\mathbf{e}_i : d+1 \leqslant i \leqslant n\}$ is a basis of $V$, and each $\mathsf{A}_i$ has the block decomposition

$$\mathsf{A}_i = \begin{bmatrix} \mathsf{I} & \mathbf{0} \\ \mathsf{C}_i & \tilde{\mathsf{A}}_i \end{bmatrix},$$

as in (4.5). The hypothesis $E_1 = E_1(\mathsf{A}_i)$ for all $i$ gives

$$E_1(\tilde{\mathsf{A}}_0) = E_1(\tilde{\mathsf{A}}_1) = \cdots = E_1(\tilde{\mathsf{A}}_{m-1}) = \mathbf{0}.$$

Since $\Sigma$ is an RCP set, so is $\Sigma' = \{\tilde{\mathsf{A}}_0, \ldots, \tilde{\mathsf{A}}_{m-1}\}$. Now by Theorem 2.2 for all $\mathbf{d} \in \mathsf{S}_m$ all the rows of the limit function $\mathsf{M}_{\Sigma'}(\mathbf{d})$ are in $E_1(\Sigma') = \mathbf{0}$; it follows that $\mathsf{M}_{\Sigma'}$ is identically zero. ■

Corollary 4.2a. *If $\Sigma$ is a finite RCP set having a continuous limit function, then all infinite products from $\Sigma$ uniformly converge to a limit matrix at a geometric rate.*

*Proof*. This follows from (4.3) and (4.7). ■

Now we treat the case of RCP sets having a real-continuous limit function.

Theorem 4.3. *The finite ordered set $\Sigma = \{\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_{m-1}\}$ of $n \times n$ matrices is an RCP set with a real-continuous limit function $\overline{\mathsf{M}}_\Sigma$ if and only if $\Sigma$ is an RCP set having a continuous limit function on $\mathsf{S}_m$, and if*

$$\mathsf{A}_i = \mathsf{S} \begin{bmatrix} \mathsf{I}_d & \mathbf{0} \\ \mathsf{C}_i & \tilde{\mathsf{A}}_i \end{bmatrix} \mathsf{S}^{-1}, \qquad 0 \leqslant i \leqslant m-1, \tag{4.9}$$

*in which* $I_d$ *is a* $d \times d$ *identity matrix with* $d = \dim(E_1(\Sigma))$, *then*

$$C_{i+1} + \tilde{A}_{i+1}(I - \tilde{A}_0)^{-1} C_0 = C_i + \tilde{A}_i (I - \tilde{A}_{m-1})^{-1} C_{m-1}, \qquad 0 \leqslant i \leqslant m-2, \tag{4.10}$$

*where* $I$ *is an* $(n-d) \times (n-d)$ *identity matrix.*

Note that by Theorem 4.2 a similarity transformation $S$ always exists such that (4.9) holds, and this corresponds to a decomposition $\mathbb{R}^n = E_1(\Sigma) + V$ for some vector space $V$. In that case the matrix identities (4.10) can be reformulated directly as in terms of $A_i$ by using projection operators $P_{E_1}$ and $P_V$ and taking $C_i = P_V A_i P_{E_1}$, $\tilde{A}_i = P_V A_i P_V$. Also note that under the conditions of Theorem 4.3, $I - \tilde{A}_0$ and $I - \tilde{A}_{m-1}$ are invertible, since $\hat{\rho}(\tilde{\Sigma}) < 1$ by Theorem 4.2.

*Proof.* The condition for an RCP set with a continuous limit function on $S_m$ to have a real-continuous limit function is simply that the two possible limit matrices for the two possible $m$-ary expansions of all terminating rationals $l/m^j$ agree. This is equivalent to the requirement that

$$A_{i+1}(A_0)^{\infty} = A_i (A_{m-1})^{\infty}, \qquad 0 \leqslant i \leqslant m-2, \tag{4.11}$$

where $(A_j)^{\infty} = \lim_{k \to \infty} (A_j)^k$. Using the decomposition (4.9), we find using (4.6) that

$$A_j^{\infty} = S \begin{bmatrix} I_d & 0 \\ C_j^{\infty} & 0 \end{bmatrix} S^{-1},$$

where

$$C_j^{\infty} := \left(I + \tilde{A}_j + \tilde{A}_j^2 + \cdots\right) C_j = \left(I - \tilde{A}_j\right)^{-1} C_j.$$

The criterion (4.10) follows from (4.11) using this formula. ∎

Remark. Suppose that $\Sigma$ is a finite RCP set having a continuous limit function. Theorems 4.1–4.3 give a procedure to verify in a finite number of steps that $\Sigma$ has this property. One first checks that $E_1(\Sigma) = E_1(A_i)$ for $0 \leqslant i \leqslant m-1$, computes a suitable $V$ and block decomposition (4.5) for $\tilde{\Sigma} = \{\tilde{A}_0, \ldots, \tilde{A}_{m-1}\}$, and successively computes $\hat{\rho}_1(\tilde{\Sigma}), \hat{\rho}_2(\tilde{\Sigma}), \ldots$. One finds that $\hat{\rho}_t(\tilde{\Sigma}) < 1$ for some $t$, and then accepts $\Sigma$.

We do not know of any finite decision procedure to prove that a finite set $\Sigma$ is *not* an RCP set having a continuous limit function. The following undecidability result, obtained by M. S. Paterson [26], suggests by analogy that there may be no such decision procedure. Call a finite set $\Sigma$ of square matrices *mortal* if some finite product of matrices in $\Sigma$ (repetitions allowed) is the zero matrix. Paterson proved that there is no recursive procedure to decide if a finite set of $3\times3$ matrices with integer entries is mortal. In particular the set $\{\Sigma : \Sigma$ has rational entries and is mortal$\}$ is recursively enumerable but not recursive.

## 5. RCP SETS HAVING AN ARBITRARY LIMIT FUNCTION

There exist RCP sets $\Sigma$ having a discontinuous limit function, e.g., add the identity matrix to any RCP set $\Sigma'$ possessing a continuous limit function. The extra complexity of RCP sets $\Sigma$ having a discontinuous limit function arises from the fact that they contain matrices having different 1-eigenspaces. The example above of adding the identity matrix already shows that convergence to the limit function is no longer exponential in the general case—by inserting many extra copies of the identity matrix in an infinite product, the convergence rate to a limit matrix can be slowed to an arbitrary degree.

We call a (finite or infinite) set of matrices $\Sigma$ *product-bounded* if there exists a finite bound $\Delta = \Delta(\Sigma)$ such that all finite products have

$$\left\| \prod_{i=1}^{k} \mathsf{M}_i \right\| \leqslant \Delta, \qquad \text{all} \quad \mathsf{M}_i \in \Sigma. \tag{5.1}$$

The main goal of this section is to characterize the class of finite RCP sets that are product-bounded. The proof of Theorem 4.2 showed that this class includes all finite RCP sets having a continuous limit function. In fact it seems possible that this class includes all finite RCP sets.

Boundedness conjecture. All finite RCP sets are product-bounded.

There are infinite RCP sets that are not product-bounded, e.g.

$$\Sigma = \left\{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} : n = 1,2,3,\dots \right\},$$

which is an RCP set because all products of two matrices in it are zero.

To discuss general RCP sets we introduce some terminology for finite products having particular 1-eigenspaces. A finite product $\mathsf{B} = \mathsf{M}_1 \cdots \mathsf{M}_k$ with all $\mathsf{M}_i \in \Sigma$ is called a *block* of $\Sigma$ if $E_1(\Sigma) = \bigcap_{j=1}^{k} E_1(\mathsf{M}_j)$ and $E_1(\Sigma) \neq \bigcap_{j=1}^{k-1} E_1(\mathsf{M}_j)$. Let $\Sigma_B$ denote the (generally infinite) set of blocks of $\Sigma$. [$\Sigma_B$ is finite if and only if $\Sigma = \{\mathsf{A}_0, \dots, \mathsf{A}_{m-1}\}$ is finite and all $E_1(\mathsf{A}_i) = E_1(\Sigma)$, in which case $\Sigma_B = \Sigma$.]

For finite product-bounded sets $\Sigma$ we can essentially reduce the study of the RCP property of $\Sigma$ to that of $\Sigma_B$.

LEMMA 5.1. *If a finite set $\Sigma$ is a product-bounded RCP set, then:*

(1) *All strict subsets of $\Sigma$ are product-bounded RCP sets.*

(2) *The set of blocks $\Sigma_B$ is a product-bounded RCP set, and $E_1(\Sigma_B) = E_1(\Sigma)$.*

(3) *Any infinite product of matrices in $\Sigma_B$ has all its rows in $E_1(\Sigma_B)$.*

*Conversely, any finite set $\Sigma$ satisfying* (1)–(3) *is a product-bounded RCP set.*

*Proof.* $\Rightarrow$: Suppose $\Sigma$ is a product-bounded RCP set. That (1), (2) hold is clear, noting by Theorem 2.1 that each block $B \in \Sigma_B$ has $E_1(B) = E_1(\Sigma)$. To show (3), suppose that $\mathsf{M}^{(\infty)} = \prod_{j=1}^{\infty} \mathsf{B}_{e_j}$ for some sequence of blocks, where $1 \leqslant e_j < \infty$. Associate to each $\mathsf{B}_{e_j}$ the set $\Sigma(e_j)$ of all $\mathsf{A}_i$'s that occur in $\mathsf{B}_{e_j}$. Then some $\Sigma''$ contained in $\Sigma$ occurs infinitely often as a $\Sigma(e_j)$ in $\mathsf{M}^{(\infty)}$. By Theorem 2.2

$$\mathsf{M}^{(\infty)}\mathsf{A}_d = \mathsf{M}^{(\infty)} \qquad \text{if} \quad \mathsf{A}_d \in \Sigma'',$$

so that the rows of $\mathsf{M}^{(\infty)}$ are in $E_1(\Sigma'')$. Now $E_1(\Sigma'') = E_1(\Sigma)$ by definition of a block, proving (3).

$\Leftarrow$: We first prove product-boundedness. Let $\Delta$ denote the maximum of the product-boundedness constants (4.1) for all $\Sigma' \subsetneqq \Sigma$ and for $\Sigma_B$. Any finite product $\mathsf{A}_{d_1} \cdots \mathsf{A}_{d_r}$ can be parsed from left to right as a product of blocks $\prod_{j=1}^{l} \mathsf{B}_{e_j}$ followed by a product $\prod_{j=k+1}^{r} \mathsf{A}_{d_j}$ not containing a block, and this second product must be contained in some $\Sigma' \subsetneqq \Sigma$, since $\bigcap_{j=k+1}^{r} E_1(\mathsf{A}_{d_j}) \neq E_1(\Sigma)$. Hence

$$\|\mathsf{A}_{d_1} \cdots \mathsf{A}_{d_r}\| \leqslant \|\mathsf{B}_{e_1} \cdots \mathsf{B}_{e_l}\| \, \|\mathsf{A}_{d_{k+1}} \cdots \mathsf{A}_{d_r}\| \leqslant \Delta^2,$$

so $\Sigma$ is product-bounded.

To prove $\Sigma$ is an RCP set, let $\mathsf{M}_\Sigma(\mathbf{d}) = \prod_{i=1}^{\infty} \mathsf{A}_{d_i}$ denote an infinite product in $\Sigma$. Proceed from left to right to parse the infinite product into blocks. We have two cases, according to whether one gets a finite number of blocks followed by an infinite string of digits never forming a block, or else one gets an infinite number of blocks.

*Case 1.* Some $\prod_{i=n+1}^{\infty} \mathsf{A}_{d_i}$ *contains no block.* In this case $\bigcap_{i=n+1}^{\infty} E_1(\mathsf{A}_{d_i}) \neq E_1(\Sigma)$, so that all these digits are contained in some $\Sigma'$ strictly contained in $\Sigma$. Then $\prod_{i=n+1}^{\infty} \mathsf{A}_{d_i}$ converges by assumption (1), whence $\prod_{i=1}^{\infty} \mathsf{A}_{d_i}$ converges.

*Case 2.* $\mathsf{M}_{\Sigma}(\mathbf{d}) = \prod_{j=1}^{\infty} \mathsf{B}_{e_j}$, *where each* $\mathsf{B}_{e_j}$ *is a block.* By assumption (2) this infinite product of $\mathsf{B}_{e_j}$ converges to a limit $\mathsf{M}^{(\infty)}$. We must show $\prod_{j=1}^{\infty} \mathsf{A}_{d_j}$ also converges to $\mathsf{M}^{(\infty)}$. Let $\mathsf{B}^{(k)} = \prod_{j=1}^{k} \mathsf{B}_{e_j}$. Take $k_0$ so large that $\|\mathsf{B}^{(k)} - \mathsf{M}^{\infty}\| \leqslant \epsilon$ for $k \geqslant k_0$. If $\mathsf{A}^{(l)} = \prod_{j=1}^{l} \mathsf{A}_{d_i}$ contains $\mathsf{B}^{(k)}$ as an initial segment and is strictly contained in $\mathsf{B}^{(k+1)}$, then if $k \geqslant k_0$ we claim that

$$\|\mathsf{A}^{(l)} - \mathsf{M}^{(\infty)}\| \leqslant \epsilon \Delta. \tag{5.2}$$

To see this write $\mathsf{A}^{(l)} = \mathsf{B}^{(k)} \mathsf{A}_{d_{n+1}} \cdots \mathsf{A}_{d_{n+r}}$, and note that $\|\mathsf{A}_{d_{n+1}} \cdots \mathsf{A}_{d_{n+r}}\| \leqslant \Delta$, since it does not form a block, so is contained in some $\Sigma' \subsetneqq \Sigma$. Then

$$\|\mathsf{A}^{(l)} - \mathsf{M}^{(\infty)}\| \leqslant \|\mathsf{B}^{(k)} - \mathsf{M}^{(\infty)}\| \, \|\mathsf{A}_{d_{n+1}} \cdots \mathsf{A}_{d_{n+r}}\| + \|\mathsf{M}^{(\infty)} \mathsf{A}_{d_{n+1}} \cdots \mathsf{A}_{d_{n+r}} - \mathsf{M}^{(\infty)}\|.$$

The first term on the right is $\leqslant \epsilon\Delta$, and the second term is zero by assumptions (2) and (3), which assert that the rows of $M^{(\infty)}$ are in $E_1(\Sigma_B) = E_1(\Sigma)$, so that

$$\mathsf{M}^{(\infty)} \mathsf{A}_i = \mathsf{M}^{(\infty)}, \qquad 0 \leqslant i < m - 1.$$

This proves (5.2), which implies that $\prod_{i=1}^{\infty} \mathsf{A}_{d_i}$ converges. ∎

Property (3) in Lemma 5.1 need not hold for an arbitrary product-bounded RCP set. Indeed,

$$\Sigma = \left\{ \mathsf{C}_n = \begin{pmatrix} 1 - \dfrac{1}{2^n} & 0 \\ 0 & 1 - \dfrac{1}{2^n} \end{pmatrix} : n \geqslant 1 \right\}$$

is a product-bounded RCP set with $E_1(\Sigma) = E_1(\mathsf{C}_n) = \mathbf{0}$, but $\prod_{n=1}^{\infty} \mathsf{C}_n$ has a nonzero limit.

Next we characterize all product-bounded RCP sets having a zero limit function.

LEMMA 5.2. *A finite or countably infinite set $\Sigma$ of matrices is a product-bounded RCP set all of whose infinite products are zero if and only if*

(1) $\hat{\rho}_1(\Sigma) = \text{Max}\{\|M\|_s : M \in \Sigma\} < \infty$,
(2) $\hat{\rho}(\Sigma) < 1$.

*Proof.* $\Leftarrow$: For some $t$, $\hat{\rho}_t(\Sigma) < 1$. Then for any finite product in $\Sigma$ we have

$$\|M_{e_1} \cdots M_{e_r}\| \leqslant \left[\max\left(1, \hat{\rho}_1(\Sigma)^{t-1}\right)\right]\left(1 - \hat{\rho}_t(\Sigma)\right)^{-1} \hat{\rho}_t(\Sigma)^{[r/t]},$$

which proves product-boundedness and shows that all products go to zero as $r \to \infty$.

$\Rightarrow$: Let $\Delta$ be the bound (5.1) for $\Sigma$, so that $\hat{\rho}_1(\Sigma) \leqslant \Delta$ proves (1). Assume, without loss of generality, that $\Delta \geqslant 1$. We prove (2) by contradiction. Suppose $\hat{\rho}(\Sigma) \geqslant 1$. Then for each $k \geqslant 1$ we can find $C_k = M_{e_{1,k}} \cdots M_{e_{k,k}}$ with $\|C_k\| \geqslant 1$. We shall construct an infinite product $M^{(\infty)} = \prod_{i=1}^{\infty} M_{f_j}$ with $M^{(i)} = \prod_{j=1}^{i} M_{f_j}$ having

$$\|M^{(i)}\| \geqslant \frac{1}{2\Delta}, \qquad i \geqslant 1. \tag{5.3}$$

This implies that $M^{(\infty)}$ is nonzero, a contradiction that will prove (2). To accomplish this construction, we define subsequences $M_{e_{j,k_i(n)}}$ as follows. Since the $M_{e_{1,k}}$ are uniformly bounded, they have a limit point $\tilde{M}_1$. There exists therefore a subsequence $M_{e_{i,k_1(n)}}$ such that, for all $n$,

$$\|M_{e_{1,k_1(n)}} - \tilde{M}_1\| \leqslant \tfrac{1}{8}\Delta^{-3}.$$

Consider now the $M_{e_{2,k_1(n)}}$. By the same argument, they have a limit point $\tilde{M}_2$, and there exists a subsequence $M_{e_{2,k_2(n)}}$ such that

$$\|M_{e_{2,k_2(n)}} - \tilde{M}_2\| \leqslant \tfrac{1}{16}\Delta^{-3}.$$

Since the $k_2(n)$ constitute a subsequence of the $k_1(n)$, we also have

$$\|\mathsf{M}_{e_{1,k_2(n)}} - \tilde{\mathsf{M}}_1\| \leqslant \tfrac{1}{8}\Delta^{-3}.$$

The construction can be repeated for every $j$, resulting in successively nested subsequences $k_l(n)$ such that, for all $n$,

$$\|\mathsf{M}_{e_{j,k_i(n)}} - \tilde{\mathsf{M}}_j\| \leqslant 2^{-j-2}\Delta^{-3}, \qquad 1 \leqslant j \leqslant i.$$

Choose now $\mathsf{M}_{f_j} = M_{e_{j,k_j(1)}}$. It follows that for every $i$ there exists $l \geqslant i$ [take e.g. $l = k_i(i)$] such that

$$\|\mathsf{M}_{f_j} - \mathsf{M}_{e_{j,l}}\| \leqslant \Delta^{-3}2^{-j-1}, \qquad 1 \leqslant j \leqslant i. \tag{5.4}$$

Now by assumption

$$\begin{aligned} 1 &\leqslant \|\mathsf{C}_l\| \leqslant \|\mathsf{M}_{e_{1,l}} \cdots \mathsf{M}_{e_{l,l}}\| \leqslant \|\mathsf{M}_{e_{1,l}} \cdots \mathsf{M}_{e_{i,l}}\| \, \|\mathsf{M}_{e_{i+1,l}} \cdots \mathsf{M}_{e_{l,l}}\| \\ &\leqslant \|\mathsf{M}_{e_{1,l}} \cdots \mathsf{M}_{e_{i,l}}\|\Delta, \end{aligned} \tag{5.5}$$

where we have again used the product-boundedness of $\Sigma$. Furthermore

$$\begin{aligned} \|\mathsf{M}^{(i)} - \mathsf{M}_{e_{1,l}} \cdots \mathsf{M}_{e_{i,l}}\| &\leqslant \sum_{j=1}^{i} \left\|\mathsf{M}^{(j-1)}\left(\mathsf{M}_{f_j} - \mathsf{M}_{e_{j,l}}\right)\mathsf{M}_{e_{j+1,l}} \cdots \mathsf{M}_{e_{i,l}}\right\| \\ &\leqslant \sum_{j=1}^{i} \|\mathsf{M}^{(j-1)}\| \|\mathsf{M}_{e_{j+1,k}} \cdots \mathsf{M}_{e_{i,k}}\|\Delta^{-3}2^{-j-1} \\ &\leqslant \Delta^{-1} \sum_{j=1}^{\infty} 2^{-j-1} \leqslant \tfrac{1}{2}\Delta^{-1}, \end{aligned}$$

using product-boundedness. Combining this with (5.5) proves (5.3). ∎

Using Lemma 5.2, we characterize finite product-bounded RCP sets in a manner analogous to that in Section 4.

THEOREM 5.1. *A finite set* $\Sigma$ *is a product-bounded RCP set of* $n \times n$ *matrices if and only if*:

(1) *All strict subsets of* $\Sigma$ *are product-bounded RCP sets.*

(2) *All* $\mathsf{B}_j \in \Sigma_B$ *have* $E_1(\mathsf{B}_j) = E_1(\Sigma)$.

(3) *There is a subspace* $V$ *of* $\mathbb{R}^n$ *such that* $E_1(\Sigma) + V = \mathbb{R}^n$, $\dim(V) = n - \dim(E_1(\Sigma))$, *and the set* $\mathsf{P}_V \Sigma_B \mathsf{P}_V = \{\mathsf{P}_V \mathsf{B} \mathsf{P}_V : \mathsf{B} \in \Sigma_B\}$, *where* $\mathsf{P}_V$ *is orthogonal projection on* $V$, *has*

$$\hat{\rho}(\mathsf{P}_V \Sigma_B \mathsf{P}_V) < 1. \tag{5.6}$$

Before proving this theorem (which parallels the proof of Theorem 4.2) we make some remarks.

(i) If conditions (1)–(3) hold for some subspace $V$, they hold for all subspaces $V$ with $E_1(\Sigma) + V = \mathbb{R}^n$, $V \cap E_1(\Sigma) = \mathbf{0}$.

(ii) The condition (5.6) implies that all infinite products of elements of $\Sigma_B$ converge at a uniform geometric rate. However, this gives no convergence bound for infinite products from $\Sigma$.

(iii) The criterion (5.6) is generally not effectively computable, since there are generally infinitely many matrices in $\mathsf{P}_V \Sigma_B \mathsf{P}_V$. Sometimes this criterion can be verified analytically.

*Proof.* $\Rightarrow$: Property (1) is clear, and (2) follows from Theorem 2.1. To verify (3) take any subspace $V$ with $E_1(\Sigma) + V = \mathbb{R}^n$, $E_1(\Sigma) \cap V = \mathbf{0}$, and without loss of generality perform a similarity transformation so that

$$\mathsf{A}_j = \begin{bmatrix} \mathsf{I} & \mathbf{0} \\ \mathsf{C}_j & \tilde{\mathsf{A}}_j \end{bmatrix}, \qquad \text{all} \quad \mathsf{A}_j \in \Sigma. \tag{5.7}$$

Then $\mathsf{P}_V \Sigma \mathsf{P}_V$ is clearly a product-bounded RCP set with

$$\mathsf{P}_V \mathsf{A}_j \mathsf{P}_V = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathsf{A}}_j \end{bmatrix}.$$

Also in this case

$$\mathsf{B}_j = \begin{bmatrix} \mathsf{I} & \mathbf{0} \\ \mathsf{D}_j & \tilde{\mathsf{B}}_j \end{bmatrix}, \qquad \text{all} \quad \mathsf{B}_j \in \Sigma_B, \tag{5.8}$$

and $\mathsf{P}_V\Sigma_B\mathsf{P}_V$ is a product-bounded RCP set. Using property (2), we have $E_1(\mathsf{P}_V\Sigma_B\mathsf{P}_V)=\mathbf{0}$. By Lemma 5.1 all infinite products from $\mathsf{P}_V\Sigma_B\mathsf{P}_V$ are zero. Now we may apply Lemma 5.2 to conclude that (3) holds.

$\Leftarrow$: Suppose that conditions (1)–(3) hold for $\Sigma$. One may by a suitable similarity transformation reduce to the case that all matrices in $\Sigma$ and $\Sigma_B$ have the forms (5.7), (5.8) respectively, where the rows of $\mathsf{I}$ are a basis of $E_1(\Sigma)$. Now we show that

$$\hat{\rho}_1(\Sigma_B)=\max\{\|\mathsf{B}\|_s:\mathsf{B}\in\Sigma_B\}<\infty. \tag{5.9}$$

Let $\Delta$ be the maximum bound for all $\Sigma'\subsetneqq\Sigma$. By definition of a block $\mathsf{B}=\mathsf{A}_{d_1}\cdots A_{d_k}$ one has $\bigcap_{i=1}^{k-1}E_1(\mathsf{A}_{d_i})\neq E_1(\Sigma)$; hence $\{\mathsf{A}_{d_1},\dots,\mathsf{A}_{d_{k-1}}\}\subseteq\Sigma'$ for some strict subset $\Sigma'$ of $\Sigma$, and $\{\mathsf{A}_{d_k}\}$ is also a strict subset of $\Sigma$, since $m\geqslant 2$. Hence

$$\|\mathsf{B}\|_s\leqslant\|\mathsf{A}_{d_1}\mathsf{A}_{d_2}\cdots\mathsf{A}_{d_{k-1}}\|_s\|\mathsf{A}_{d_k}\|_s\leqslant\Delta^2.$$

This proves $\hat{\rho}_1(\Sigma_B)\leqslant\Delta^2$, proving (5.9).

Next, since $\|\tilde{\mathsf{B}}_j\|_s\leqslant\|\mathsf{B}_j\|_s$ in (5.8), we have $\hat{\rho}_1(\mathsf{P}_V\Sigma_B\mathsf{P}_V)\leqslant\hat{\rho}_1(\Sigma_B)<\infty$. Together with (3) and Lemma 5.2, this implies that $\mathsf{P}_V\Sigma_B\mathsf{P}_V$ is a bounded RCP set all of whose infinite products are zero. Now note that (5.9) gives $\|\mathsf{P}_j\|_s\leqslant\rho_1(\Sigma_B)\leqslant\Delta^2$. One has

$$\mathsf{B}_{j_1}\cdots\mathsf{B}_{j_r}=\begin{bmatrix}\mathsf{I}&\mathbf{0}\\ \tilde{\mathsf{D}}&\tilde{\mathsf{B}}\end{bmatrix}, \tag{5.10}$$

where $\tilde{\mathsf{B}}=\tilde{\mathsf{B}}_{j_1}\cdots\tilde{\mathsf{B}}_{j_r}$ and $\tilde{\mathsf{D}}=\Sigma_{i=1}^r\tilde{\mathsf{B}}_{j_1}\cdots\tilde{\mathsf{B}}_{j_{i-1}}\mathsf{D}_{j_i}$. By property (3) there exists some $t$ with $\hat{\rho}_t(\mathsf{P}_V\Sigma_B\mathsf{P}_V)<1$, which gives the bounds

$$\|\tilde{\mathsf{B}}\|_s=\|\tilde{\mathsf{B}}_{j_1}\cdots\tilde{\mathsf{B}}_{j_m}\|_s\leqslant\Delta^{2(t-1)}\hat{\rho}_t(\mathsf{P}_V\Sigma\mathsf{P}_V)^{[r/t]} \tag{5.11}$$

and

$$\|\tilde{\mathsf{D}}\|_s\leqslant\Delta^{2(t-1)}(1-\hat{\rho}_t(\mathsf{P}_V\Sigma_B\mathsf{P}_V))^{-1}\hat{\rho}_t(\mathsf{P}_V\Sigma\mathsf{P}_V)^{[r/t]}. \tag{5.12}$$

Now we have the bound

$$\|\mathsf{B}_{j_1}\cdots\mathsf{B}_{j_r}\|_s\leqslant\|\mathsf{I}\|_s+\|\tilde{\mathsf{D}}\|_s+\|\tilde{\mathsf{B}}\|_s\leqslant 1+\|\tilde{\mathsf{D}}\|_s+\|\tilde{\mathsf{B}}\|_s,$$

which with (5.11), (5.12) shows that $\Sigma_B$ is product-bounded, and also implies that $\Sigma_B$ is an RCP set.

Let $\tilde{\Delta}$ be a bound for $\Sigma_B$. Then for any $r$ one has

$$\|\mathsf{A}_{d_1} \cdots \mathsf{A}_{d_r}\|_s \leqslant \Delta\tilde{\Delta},$$

by parsing this product into a set of blocks in $\Sigma_B$, followed by a collection of $\mathsf{A}_i$'s in some proper subset $\Sigma'$ of $\Sigma$. This shows that $\Sigma$ is product-bounded.

Finally one shows that $\Sigma$ is an RCP set by considering two cases as in Lemma 5.2. We omit the details, observing only that property (2) is used in proving that the limits $\mathsf{M}^{(\infty)}$ have all rows in $E_1(\Sigma) = E_1(\Sigma_B)$. ∎

## 6. RCP SETS AND LIMIT FUNCTIONS: EXAMPLES

RCP sets of matrices arise in several different areas. Long products of nonnegative matrices arise in connection with nonhomogeneous Markov chains [16], mathematical demography [4], probabilistic automata [28], and random walks in a random environment [19]. The RCP property is a special kind of "ergodic" property of such products, stronger than the usual ones. RCP sets arise also in deterministic constructions of functions and curves with self-similarities under changes in scale. For example, we describe below how the Koch snowflake curve can be parametrized using the limit function of a particular RCP set. The construction of wavelets of compact support (see [7–9]) uses solutions of two-scale difference equations (see introduction), and these are described by limit functions of RCP sets, as are the dyadic interpolation schemes of Deslauriers and Dubuc [10–12]. In these examples the RCP property is generally proved by special methods, not by appeal to the general results of this paper. All the examples below have real-continuous limit functions. There exist examples (arising in [8, 9]) of limit functions of class $C^k$ for arbitrary finite $k$. It is clear from the second set of examples that various fractal curves can be obtained as limit functions of RCP sets.

EXAMPLE 1 (Nonhomogeneous Markov chains). A nonhomogeneous Markov chain is one whose transition probabilities may change after each step; see [35, 36]. Mathematically their study corresponds to the study of products of arbitrary stochastic matrices. To be consistent with earlier notation we treat Markov-chain matrices as *column-stochastic matrices*, which are nonnegative matrices with all column sums equal to one. Such matrices have $\mathbf{e} = (1, 1, \ldots, 1)$ as a left eigenvector with eigenvalue 1. (The Markov-chain literature generally uses row-stochastic matrices, so all concepts defined here are transposes of the usual ones.) The product of column-stochastic matrices is column-stochastic, which implies that any set $\Sigma$ of column-stochastic matrices is product-bounded in the sense of Section 4.

A column-stochastic matrix $\mathsf{P}$ is *regular* if it has exactly one eigenvalue equal to 1 and all other eigenvalues are less than 1 in absolute value. Two equivalent notions of regularity are that the Markov chain associated to $\mathsf{P}$ is irreducible and aperiodic, or that $\lim_{k\to\infty}\mathsf{P}^k$ has rank one, in which case all its rows are equal. Hajnal [16] observes that the product $\mathsf{P}_1\mathsf{P}_2$ of two regular matrices need not be regular. He introduced the notion of a *scrambling matrix*, which is one in which each pair of columns have positive entries in some common row. Scrambling matrices are regular, and have the property that the product $\mathsf{P}_1\mathsf{P}_2$ is scrambling if one of $\mathsf{P}_1$ and $\mathsf{P}_2$ is. Hajnal [16] also introduced the concept of *weak ergodicity* of a left-infinite product of column stochastic matrices. The infinite product $\prod^L \mathsf{P}_r$ is weakly ergodic if

$$\mathsf{M}^{(r,s)} = \mathsf{P}_{r+s}\mathsf{P}_{r+s-1}\cdots\mathsf{P}_{r+1}\mathsf{P}_r$$

satisfies for all $r \geqslant 1$, and all $i, j, k$,

$$\lim_{s\to\infty} \mathsf{M}_{ij}^{(r,s)} - \mathsf{M}_{ik}^{(r,s)} = \mathbf{0}.$$

That is, the rows of $\mathsf{M}^{(r,s)}$ approach constant rows as $s \to \infty$, but the constants vary as $r$ varies and the left-infinite product $\prod^L \mathsf{P}_r$ need not converge.

The results of several authors combine to give a complete characterization of finite RCP sets of column-stochastic matrices containing a regular matrix.

THEOREM 6.1. *For a finite set* $\Sigma = \{\mathsf{P}_0, \mathsf{P}_1, \ldots, \mathsf{P}_{m-1}\}$ *of* $n \times n$ *column-stochastic matrices, the following conditions are equivalent*:

(C1) $\Sigma$ *is an RCP set whose left* 1*-eigenspace* $E_1(\Sigma)$ *is one-dimensional, spanned by* $(1, 1, \ldots, 1)$.

(C2) *All finite products* $\mathsf{P}_{d_1}\cdots\mathsf{P}_{d_k}$ *are irreducible and aperiodic.*

(C3) *There exists a finite* $s$ *such that for all* $k \geqslant s$ *all products* $\mathsf{P}_{d_1}\cdots\mathsf{P}_{d_k}$ *are scrambling.*

(C4) *There exists a finite* $\mu$ *such that for all* $k \geqslant \mu$ *all* $\mathsf{P}_{d_1}\cdots\mathsf{P}_{d_k}$ *have a row with all entries nonzero.*

(C5) *All left-infinite products from* $\Sigma$ *are weakly ergodic.*

*Proof.* The equivalence of (C2), (C3), and (C4) appears in Anthonisse and Tijms [1], though some of it is due to [16, 35, 38]. The implication (C1) $\Rightarrow$ (C4) follows, since then $(\mathsf{P}_{d_1} \cdots \mathsf{P}_{d_k})^m$ converges to a limit in $E_1(\Sigma)$ as $m \to \infty$. To show (C2),(C4) $\Rightarrow$ (C1) observe first that all $E_1(\Sigma)$ and all $E_1(\mathsf{P}_i)$ are one-dimensional, since $\mathsf{P}_i$ is irreducible and aperiodic by (C2). Then $\Sigma$ is an RCP set by Theorems 4.1 and 4.2, because (C4) implies that $\hat{\rho}_s(\Sigma') < 1$ for $\Sigma' = \mathsf{P}_V \Sigma \mathsf{P}_V$, where $V$ is the orthogonal complement of $E_1(\Sigma)$. The implication (C1) $\Rightarrow$ (C5) follows from the uniform exponential convergence rate stated in Corollary 4.2a. (C5) $\Rightarrow$ (C4) follows by examining $(\mathsf{P}_{d_1} \cdots \mathsf{P}_{d_k})^m$ as $m \to \infty$ as a left-infinite product, and observing that it approaches a rank-1 matrix.

This proof actually shows that any $\Sigma$ satisfying (C1)–(C5) has a continuous limit function, and all limit matrices have constant rows. ∎

The implication (C4) $\Leftrightarrow$ (C1) was also obtained by Micchelli and Prautzsch [23, Theorem 2.1]. Paz [27] shows that if (C4) holds, then it holds with $\mu \leqslant \frac{1}{2}(3^n - 2^{n+1} + 1)$ and this bound is sharp. This implies the following result.

COROLLARY 6.1a. *There is an effectively computable procedure to decide whether or not a finite set $\Sigma$ of column-stochastic matrices with rational entries is an RCP set with* $\dim(E_1(\Sigma)) = 1$.

Hartfiel [18] gives some sufficient conditions for infinite products of nonnegative matrices to converge to the zero matrix. Several authors [1, 18, 36] have observed that in this case there is an exponential rate of convergence, which is a special case of Corollary 4.2a.

EXAMPLE 2 (Koch snowflake curve and de Rham curves). The Koch snowflake curve [37] is a continuous, nowhere differentiable curve constructed iteratively as indicated in Figure 1. There is a natural parametrization $(x(\alpha), y(\alpha))$ of the limit curve for $\alpha \in [0,1]$ obtained by associating to $(x,y)$ the sequence of nested intervals to which $(x,y)$ belongs at each iteration, labeled 0, 1, 2, and 3 in Figure 2, and assigning that $\alpha$ whose base-4 expansion is given by this sequence. This procedure has the ambiguity that points $(x,y)$ that are endpoints of some interval have two sequences of nested intervals they belong to, but the two distinct base-4 expansions that result give the same real number $\alpha$, which is rational.

One can carry out this construction by keeping track of the endpoints of the interval. Let $(x_n^-, y_n^-)$ denote the left endpoint of the interval at the $n$th

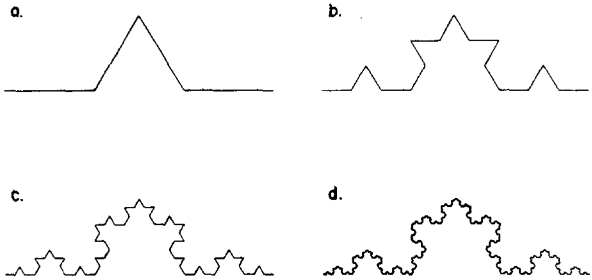


FIG. 1. Successive iterative steps (labeled by $n$) of the construction of the Koch snowflake curve: (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, (d) $n = 5$. The last iteration is already indistinguishable, at this scale, from the limit as $n$ tends to $\infty$.

iterative step, and $(x_n^+, y_n^+)$ the right endpoint. If one takes subinterval $j$ for $0 \leqslant j \leqslant 3$ at the $(n+1)$st iteration, then

$$(x_n^-, y_n^-, x_n^+, y_n^+)\mathbf{M}_j = (x_{n+1}^-, y_{n+1}^-, x_{n+1}^+, y_{n+1}^+), \tag{6.1}$$

where the $\mathbf{M}_j$ represent homogeneous linear transformations indicated in

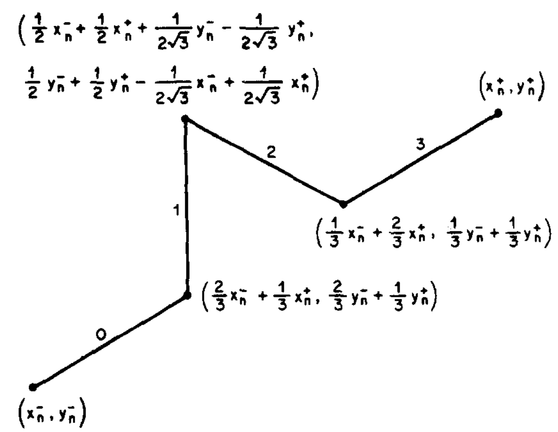


FIG. 2. The iterative construction of the Koch snowflake curve: the four intervals of the $(n+1)$th iteration step (labeled 0, 1, 2, and 3) obtained from the $n$th step interval with endpoints $(x_n^-, y_n^-)$ and $(x_n^+, y_n^+)$.

Figure 2, given by

$$\mathsf{M}_0=\begin{bmatrix}1&0&\frac{2}{3}&0\\0&1&0&\frac{2}{3}\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{3}\end{bmatrix},\qquad \mathsf{M}_1=\begin{bmatrix}\frac{2}{3}&0&\frac{1}{2}&-\frac{1}{6}\sqrt{3}\\0&\frac{2}{3}&\frac{1}{6}\sqrt{3}&\frac{1}{2}\\\frac{1}{3}&0&\frac{1}{2}&\frac{1}{6}\sqrt{3}\\0&\frac{1}{3}&-\frac{1}{6}\sqrt{3}&\frac{1}{2}\end{bmatrix},$$

$$\mathsf{M}_2=\begin{bmatrix}\frac{1}{2}&-\frac{1}{6}\sqrt{3}&\frac{1}{3}&0\\\frac{1}{6}\sqrt{3}&\frac{1}{2}&0&\frac{1}{3}\\\frac{1}{2}&\frac{1}{6}\sqrt{3}&\frac{2}{3}&0\\-\frac{1}{6}\sqrt{3}&\frac{1}{2}&0&\frac{2}{3}\end{bmatrix},\qquad \mathsf{M}_3=\begin{bmatrix}\frac{1}{3}&0&0&0\\0&\frac{1}{3}&0&0\\\frac{2}{3}&0&1&0\\0&\frac{2}{3}&0&1\end{bmatrix}.$$

The initial condition is $(x_0^-,y_0^-,x_0^+,y_0^+)=(0,0,1,0)$.

THEOREM 6.2. *$\Sigma=\{\mathsf{M}_0,\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3\}$ is an RCP set with* $\dim(E_1(\Sigma))=2$. *The vectors* $(0,1,0,1)$ *and* $(1,0,1,0)$ *constitute a basis for $E_1(\Sigma)$. The real-limit function is*

$$\overline{\mathsf{M}}_\Sigma(\alpha)=\begin{bmatrix}1-x(\alpha)&-y(\alpha)&1-x(\alpha)&-y(\alpha)\\y(\alpha)&1-x(\alpha)&y(\alpha)&1-x(\alpha)\\x(\alpha)&y(\alpha)&x(\alpha)&y(\alpha)\\-y(\alpha)&x(\alpha)&-y(\alpha)&x(\alpha)\end{bmatrix},$$

*where $(x(\alpha),y(\alpha))$ parametrizes the Koch snowflake curve.*

*Proof.* It is clear using (6.1) that

$$\lim_{k\to\infty}(0,0,1,0)\mathsf{M}_{d_1}\mathsf{M}_{d_2}\cdots\mathsf{M}_{d_k}$$

exists and equals $(x(\alpha),y(\alpha),x(\alpha),y(\alpha))$, so the third row of the limit function exists. The same construction starting with $(1,0,0,0)$ in (6.1) gives the Koch curve reversed and reflected about the $x$-axis, whence

$$\begin{aligned}\lim_{k\to\infty}(1,0,0,0)\mathsf{M}_{d_1}\cdots\mathsf{M}_{d_k}&=(x(1-\alpha),-y(\alpha),x(1-\alpha),-y(\alpha))\\&=(1-x(\alpha),-y(\alpha),1-x(\alpha),-y(\alpha)),\end{aligned}$$

using an obvious symmetry of the Koch curve. So the first row of $\overline{\mathsf{M}}_\Sigma(\alpha)$ exists. Similarly one shows the other two rows exist, so $\overline{\mathsf{M}}_\Sigma(\alpha)$ exists and $\Sigma$ is an RCP set. The assertions about $E_1(\Sigma)$ follow by an easy calculation. ∎

de Rham [30–32] studied curves constructed by a similar iterative process, and observed that they could be described by composing certain nonhomogeneous linear transformations in two variables. These can be encoded by RCP sets of $3\times 3$ matrices, e.g., the curve of [30] has the associated RCP set

$$\Sigma = \left\{ \mathsf{M}_0 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{M}_1 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

with limit function

$$\begin{bmatrix} 0 & 0 & x(\alpha) \\ 0 & 0 & y(\alpha) \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 3 (Two-scale difference equations). *Two-scale difference equations* are functional equations of the form

$$f(x) = \sum_{j=-N}^{N} a_j f(kx + j),$$

where $k \geqslant 2$ is integral. Such equations (with $k = 2$) arise in the construction of wavelets of compact support [7] and in the dyadic interpolation schemes of Deslauriers and Dubuc [10–13]. A *wavelet of compact support* $g(x)$ is a compactly supported complex-valued function on $\mathbb{R}$ such that $\{g(2^k x + l)\colon k \geqslant 0,\ l \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. In [8, 9] it is shown that solutions of two-scale difference equations can be constructed from the limit functions of certain associated RCP sets. For example, the two-scale difference equation

$$f(x) = \frac{1+\sqrt{3}}{4} f(2x) + \frac{3+\sqrt{3}}{4} f(2x+1) + \frac{3-\sqrt{3}}{4} f(2x+2) + \frac{1-\sqrt{3}}{4} f(2x+3),$$

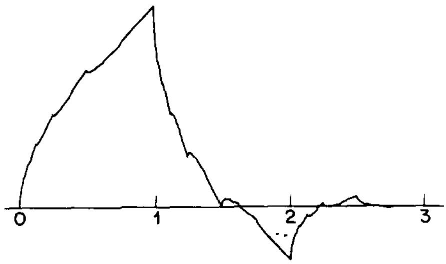


FIG. 3. The $L^1$-solution of the two-scale difference equation $f(x) = \frac{1}{4}(1+\sqrt{3})f(2x)+\frac{1}{4}(3+\sqrt{3})f(2x-1)+\frac{1}{4}(3-\sqrt{3})f(2x-2)+\frac{1}{4}(1-\sqrt{3})f(2x-3)$.

has a unique solution (up to scaling by a constant factor) in $L^1(\mathbb{R})$, and this solution is supported on the interval $[0,3]$. (See Figure 3.)

In [8] it is shown that the set $\Sigma = \{\mathbf{M}_0, \mathbf{M}_1\}$ given by

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1+\sqrt{3}}{4} & 0 & 0 \\ \frac{3-\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ 0 & \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} \end{bmatrix}$$

$$\mathbf{M}_1 = \begin{bmatrix} \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} & 0 \\ \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \\ 0 & 0 & \frac{1-\sqrt{3}}{4} \end{bmatrix}$$

is an RCP set with $E_1(\Sigma)$ generated by $(1,1,1)$. It has real-limit function

$$\overline{\mathbf{M}}_\Sigma(\alpha) = \begin{bmatrix} f(\alpha) & f(\alpha) & f(\alpha) \\ f(\alpha+1) & f(\alpha+1) & f(\alpha+1) \\ f(\alpha+2) & f(\alpha+2) & f(\alpha+2) \end{bmatrix}$$

for $0 \leqslant \alpha \leqslant 1$. The function $f$ is continuous and is differentiable on $[0,3]$ except on a set of points of Hausdorff dimension $\leqslant 0.25$. The methods of [8–9] permit one to construct examples of RCP sets whose limit functions are in $\mathbf{C}^k([0,1])$ for any fixed $k$.

## 7. PRODUCTS OF RANDOM MATRICES

The results of this paper parallel certain results on products of random matrices, on which there is a large literature [6]. Furstenberg and Kesten [15] showed that products of random matrices generated by a stationary process have, under general conditions, an asymptotic growth rate, which is analogous to a logarithm of a spectral radius. This result was greatly extended and sharpened by Oseledec [25] (see also [19, 29]), who proved under similar hypotheses the existence of $n$ constants (called *Lyapunov numbers*) analogous to logarithms of eigenvalues, the largest of which is the asymptotic growth rate above.

Oseledec's multiplicative ergodic theorem. *Let $\{\mathbf{M}_k\}_{k \geqslant 1}$ be a stationary ergodic sequence of $n \times n$ real matrices on the probability space $(\Omega, \mathbf{B}, m)$, and let $\mathbf{P}_k = \mathbf{M}_1\mathbf{M}_2 \cdots \mathbf{M}_k$. Suppose that $E[\log^+ \|\mathbf{M}_0\|] < \infty$, where* $\log^+ x = \max(0, \log(|x|)$. *Then there are n constants $-\infty \leqslant d_1 \leqslant \cdots \leqslant d_n < \infty$ and a strictly increasing nonrandom sequence of integers*

$$1 = i_1 < i_2 < \cdots < i_p < i_{p+1} = n+1$$

*which mark the points of increase in the values of the $d_i$, such that for almost every $\omega \in \Omega$ the following hold*:

(1) *For all* $\mathrm{v} \in \mathbb{R}^n$, $\lim_{k \to \infty}(1/k)\log\|\mathrm{v}\mathbf{P}_k\|$ *exists or is* $-\infty$.

(2) *For* $1 \leqslant q \leqslant p$,

$$V(q,\omega) = \left\{ \mathrm{v} \in \mathbb{R}^n : \lim_{k \to \infty} \frac{1}{n} \log\|\mathrm{v}\mathbf{P}_k\| \leqslant d_{i_q} \right\}$$

*is a random vector subspace of $\mathbb{R}^n$ of dimension $i_{q+1} - 1$.*

(3) *If $V(0,\omega)$ denotes* $\mathbf{0}$, *then for $1 \leqslant q \leqslant p$ and* $\mathrm{v} \in V(q,\omega) - V(q-1,\omega)$,

$$\lim_{k \to \infty} \frac{1}{k} \log\|\mathrm{v}\mathbf{P}_k\| = d_{i_q}.$$

If the $\{\mathsf{M}_k\}$ are i.i.d., then one has

$$\lim_{k\to\infty}\frac{1}{k}\log\|\mathsf{P}_k\| = d_n$$

almost surely.

Finite RCP sets $\Sigma$ behave like products of random matrices that have exactly $\dim(E_1(\Sigma))$ of their Lyapunov exponents equal to zero, and whose remaining Lyapunov exponents are negative. In fact, if one associates to $\Sigma = \{\mathsf{A}_0,\dots,\mathsf{A}_{m-1}\}$ an i.i.d. stationary process that one gets by selecting a matrix from $\Sigma$ with a fixed probability distribution with all probabilities positive, then the Lyapunov exponents for this process given by Oseledec's theorem have this property. However, the RCP condition is more restrictive than the conclusion of Oseledec's theorem in that it applies to *all* infinite products and not just "almost all" products.

There is a relation between the largest Lyapunov exponent in Oseledec's theorem and the joint spectral radius. If we take an i.i.d. stationary process on an arbitrary finite set $\Sigma$ of real matrices, then one always has the inequality

$$d_n \leqslant \log\hat{\rho}(\Sigma),$$

where $\hat{\rho}(\Sigma)$ is the joint spectral radius.

The results of this paper parallel certain results on products of nonnegative matrices. Hajnal [17] observed that, under suitable general conditions, products of $n\times n$ nonnegative matrices converge in the following sense to rank-one matrices. If $\mathsf{M}^{(k)} = \mathsf{M}_1\cdots\mathsf{M}_k$, then for $1\leqslant j\leqslant n$ one has

$$\frac{\mathsf{M}_{ij}^{(k)}}{\mathsf{M}_{lj}^{(k)}}\to\alpha_{il}\qquad\text{as}\quad k\to\infty, \tag{7.1}$$

in which the limiting values $\alpha_{il}$ depend on the infinite product. In consequence the rows of $\mathsf{M}^{(k)}$ become proportional in the limit and $\alpha_{il} = \alpha_i/\alpha_l$ for some quantities $(\alpha_1,\alpha_2,\dots,\alpha_n)$. The Coale-Lopez theorem of mathematical demography is a result of this kind; see [4, 5]. The property (7.1) defines a kind of limit function (of ratios) for such infinite products.

The set

$$\Sigma = \left\{\begin{pmatrix}0 & 1\\ 1 & n\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n : n\geqslant 1\right\}$$

arising from the ordinary continued-fraction algorithm has the property that all infinite products from $\Sigma$ satisfy (7.1).

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