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Cohomology of Compactifications of Locally Symmetric Spaces
I. Introduction and Intersection Cohomology

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Plan of Talks

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Talk 1: Introduction and Intersection Cohomology

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Talk 4: Micro-purity of Intersection Cohomology and Functoriality of Micro-support

Introductory Example

$D \;:= \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) = \mathrm{GL}_2(\mathbb{R})/\,\mathrm{O}(2)\mathbb{R}^+$

$\cong$ the complex upper half-plane with Poincaré metric

$\Gamma \;:=$ finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$

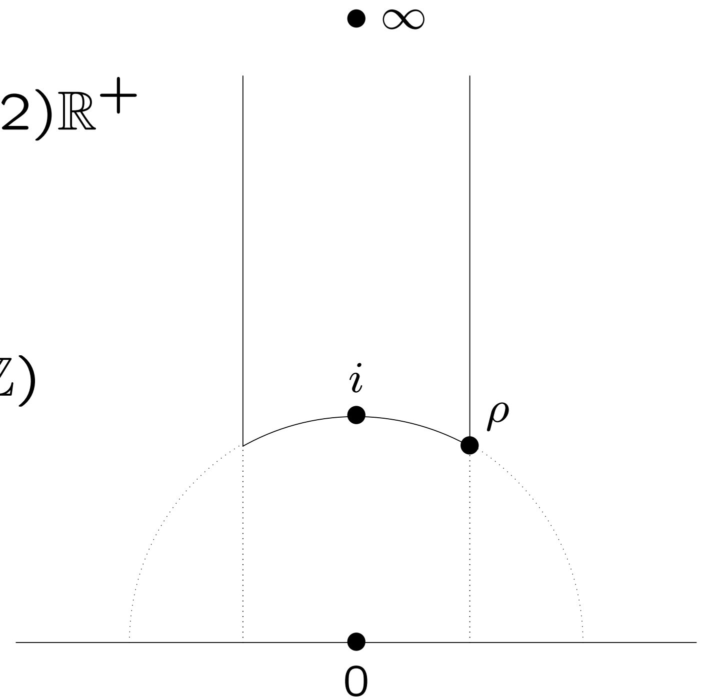
$X \;:= \Gamma\backslash D$

$\overline{D} \;:= D \cup \mathbb{R} \cup \{\infty\}$

$D^* \;:= D \cup \mathbb{Q} \cup \{\infty\}$

$X^* \;:= \Gamma\backslash D^* = X \cup \{\text{cusps}\}$, an algebraic variety (projectively embedded via modular forms)

$\mathbb{E} \;:=$ local coefficient system associated to the k^{th} symmetric power of the standard representation



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Heuristic argument. The metric near a cusp is

$$dr^2 + e^{-2r}d\theta^2 \qquad (r = \log y \in [b,\infty) \text{ and } \theta = x \mod 2\pi).$$

Thus $|d\theta|^2 dV \sim e^r dr d\theta$ and hence $d\theta$ is not L^2.

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Applications of Eichler-Shimura

(not discussed here)

- An integral structure on E yields one on $\mathcal{S}_{k+2}(\Gamma)$;

- The Hecke algebra acts on both sides and the isomorphism is one of Hecke modules;

- The space X^* is actually an algebraic variety defined over a number field. For Γ a congruence subgroup, the isomorphism allows one to relate the Hasse-Weil zeta function of X^* (which encodes the number of points of X^* defined over all finite fields) to the L-functions associated to modular forms.

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• $H^1_P(X^*;\mathbb{E}) \cong I_pH^1(X^*;\mathbb{E})$, Goresky and MacPherson's **middle-perversity intersection cohomology** (1980, 1983).

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(new version) $$I_pH^1(X^*;\mathbb{E}) \cong H^1_{(2)}(X;\mathbb{E})$$

In this form it can be generalized . . .

Locally Symmetric Spaces

A locally symmetric space has the form

$$X := \Gamma\backslash G/KA_G = \Gamma\backslash D$$

where

$$\begin{aligned} G &:= \text{the real points of a connected reductive group}/\mathbb{Q}, \\ K &:= \text{a maximal compact subgroup of } G, \\ A_G &:= \text{the identity component } (\mathbb{R}^+)^s \text{ of a maximal} \\ &\qquad \mathbb{Q}\text{-split torus in the center of } G, \\ D &:= G/KA_G, \text{ the corresponding symmetric space,} \\ \Gamma &:= \text{an arithmetic subgroup of } G. \end{aligned}$$

In our example, $X = \Gamma\backslash \mathrm{GL}_2(\mathbb{R})/\,\mathrm{O}(2)\mathbb{R}^+$.

We give D a G-invariant metric and let X have the induced metric.

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In 1982 Zucker conjectured the following theorem:

Theorem (Looijenga (1988), S. and Stern (1987, 1990)). *Let X be a Hermitian locally symmetric space, let $\mathbb{E}$ be a local coefficient system associated to a representation of G, and let p be a middle perversity. There is a natural isomorphism $I_pH(X^*;\mathbb{E}) \cong H_{(2)}(X;\mathbb{E})$.*

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One approach (Goresky-MacPherson):

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A Problem: local calculations with intersection cohomology on X^* are hard since the links of strata are complicated.

Links and strata of X^*

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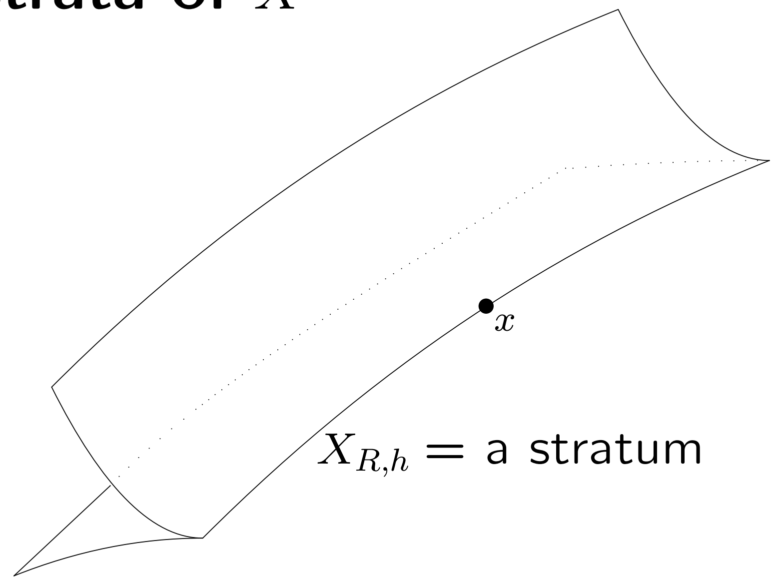

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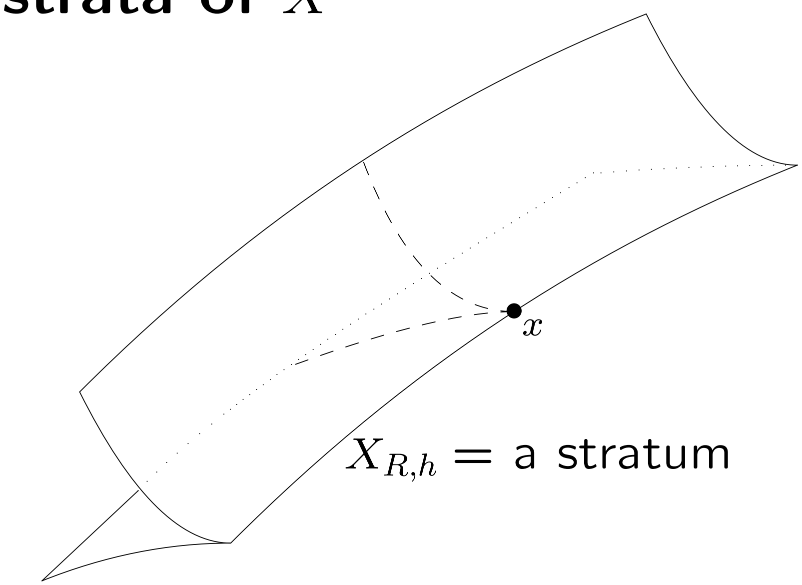

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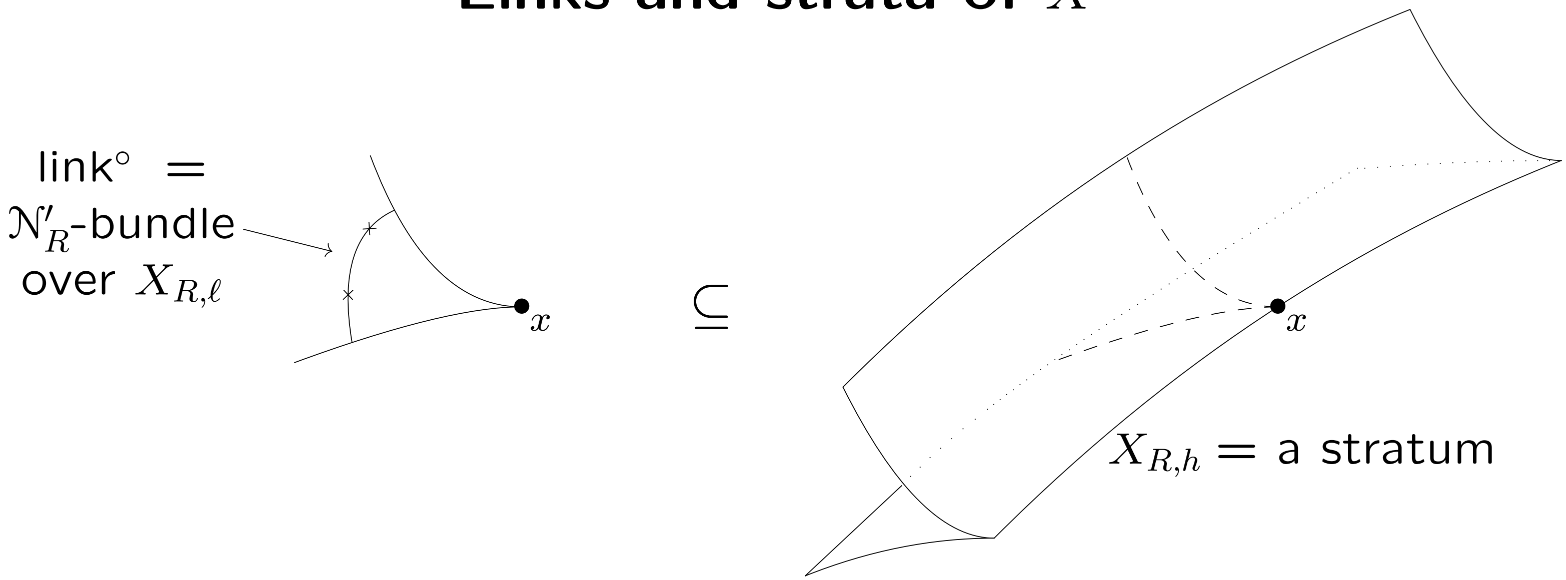

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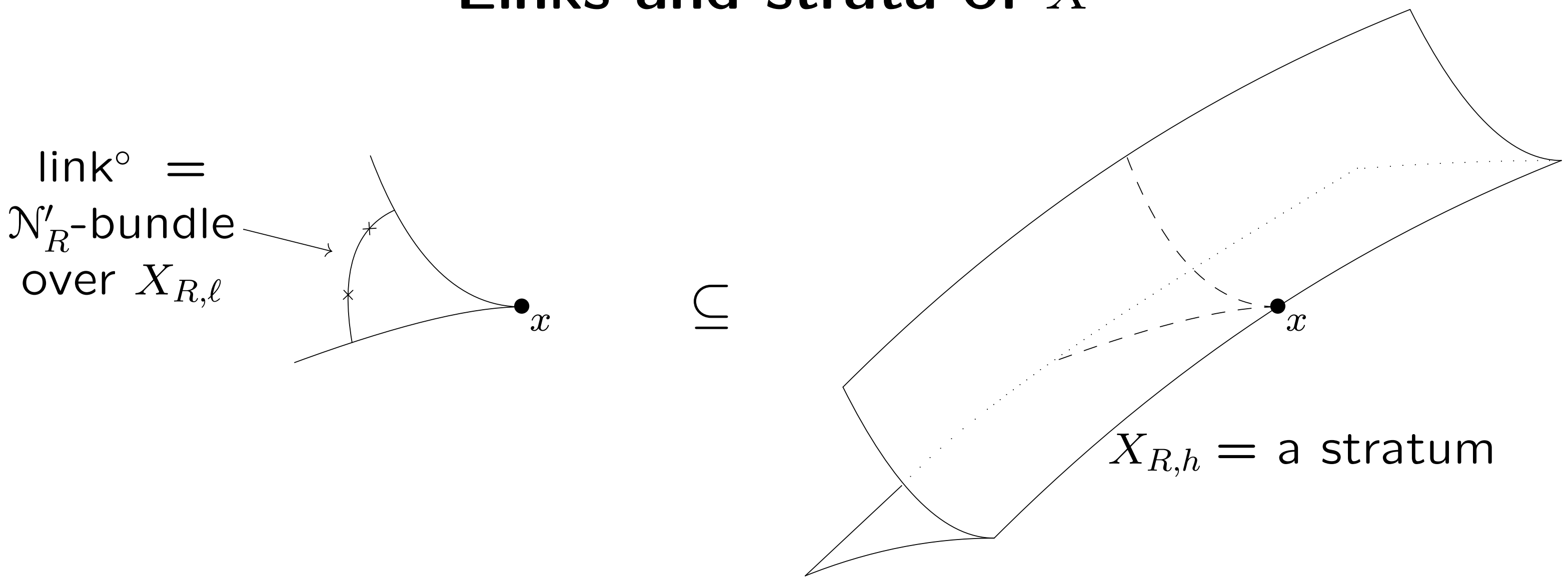

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Since we do not have an effective method in general to compute the cohomology of a locally symmetric space, the local intersection cohomology of X^* is difficult to work with.

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(i) The *toroidal compactification* $\widetilde{X} \to X^*$ (Mumford, *et al.* (1975)

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- **Conjecture** (Rapoport (1986), Goresky-MacPherson (1988)): $I_pH(X^*;\mathbb{E}) \cong I_pH(\widehat{X};\mathbb{E})$.

The Main Result

A symmetric space D is **equal-rank** if it may be written as G/K where $\mathbb{C}$-rank $G(\mathbb{C})$ = rank K. All Hermitian symmetric spaces are equal-rank.

Theorem (S., 2001). *Let X be a equal-rank locally symmetric space and let X^* be a Satake compactification for which all real boundary components are equal-rank. Let p be a middle perversity. There is a natural quasi-isomorphism $\mathcal{I}_p\mathcal{C}(X^*;\mathbb{E}) \cong \pi_*\mathcal{I}_p\mathcal{C}(\widehat{X};\mathbb{E})$ and hence an isomorphism $I_pH(X^*;\mathbb{E}) \cong I_pH(\widehat{X};\mathbb{E})$.*

This implies the Goresky-MacPherson-Rapoport conjecture.

Previous results:

- Zucker: $G = \mathrm{Sp}_4(\mathbb{R})$ and $\mathbb{E} = \mathbb{C}$.
- Goresky and MacPherson (1988, unpublished): $G = \mathrm{Sp}_4(\mathbb{R})$, $\mathrm{Sp}_6(\mathbb{R})$, and $\mathrm{Sp}_8(\mathbb{R})$ with $\mathbb{E} = \mathbb{C}$ in last case.
- S.-Stern (1992): Hermitian $\mathbb{Q}$-rank $G = 1$ case.

$\mathcal{L}$-modules and micro-support

Intersection cohomology is obtained as the hypercohomology of a constructible complex of sheaves.

An $\mathcal{L}$-**module** on $\widehat{X}$ is a combinatorial analogue of a constructible complex of sheaves on $\widehat{X}$. It has a combinatorial invariant called the **micro-support**.

The proof of the main result follows from 3 theorems:

- a **Vanishing Theorem** for cohomology of an $\mathcal{L}$-module in terms of micro-support;
- a **Micro-purity Theorem** for $\mathcal{I}_p\mathcal{C}(\widehat{X}; E)$;
- a **Functoriality Theorem** for micro-support.

These will be discussed in Talks 3 and 4.

Other applications of $\mathcal{L}$-modules: ordinary cohomology

The ordinary cohomology $H(X;E)$ is given as the cohomology of an $\mathcal{L}$-module and the vanishing theorem applies.

Theorem (S. (2001, 2003), Li and Schwermer (to appear in Duke Math. J.)). *If X is a locally symmetric space and E has regular highest weight, then*

$$H^i(X;E) = 0 \qquad \text{for } i < \frac{1}{2}(\dim X - \operatorname{rank} G(\mathbb{C}) + \operatorname{rank} K + \operatorname{rank} A_G).$$

This answers a question of Tilouine in the Hermitian case. For the case $G = R_{k/\mathbb{Q}}\mathrm{GSp}_4$ where k is a totally real number field the theorem was proven in Tilouine and Urban (1999) using results of Franke (1998). For applications of the theorem see Mauger's thesis (2000) and Mokrane and Tilouine (2000).

Other applications of $\mathcal{L}$-modules: weighted cohomology

Goresky, Harder, and MacPherson (1994) explored a different path. They work with $\widehat{X}$ but instead of intersection cohomology they define the **weighted cohomology** $W^\eta H(\widehat{X};\mathbb{E})$, an differential geometric analogue of L^2-cohomology. Here η is a **weight profile**.

Theorem (Goresky, Harder, and MacPherson (1994)). *Let X be a Hermitian locally symmetric space. Let p be a middle perversity and η a middle weight profile. There is a natural quasi-isomorphism $\mathcal{I}_p\mathcal{C}(X^*;\mathbb{E}) \cong \pi_*\mathcal{W}^\eta\mathcal{C}(\widehat{X};\mathbb{E})$ and hence an isomorphism $I_pH(X^*;\mathbb{E}) \cong W^\eta H(\widehat{X};\mathbb{E})$.*

With $\mathcal{L}$-modules we can prove a generalization of the theorem for equal-rank Satake compactifications (S., 2001).

In the Hermitian case, Goresky and MacPherson (2001) have used the isomorphism $I_pH(X^*;\mathbb{E}) \cong W^\eta H(\widehat{X};\mathbb{E})$ to prove a topological version of Arthur's trace formula on L^2-cohomology (1989); in collaboration with Kottwitz (1997) they prove that their topological trace formula agree with Arthur's formula.

Other applications of $\mathcal{L}$-modules: global isomorphisms

For X equal-rank, the previous theorems imply that $I_pH(\widehat{X};\mathbb{E}) \cong W^\eta H(\widehat{X};\mathbb{E}) \cong I_pH(X^*;\mathbb{E})$.

Even if X is not equal-rank and X^* does not exist, we can still prove the first isomorphism:
Theorem. *If $E^* \cong \overline{E}$ and the $\mathbb{Q}$-root system of G does not have any factor of type D_n, E_n, or F_4, then $I_mH(\widehat{X};\mathbb{E}) \cong W^\mu H(\widehat{X};\mathbb{E})$ and $I_nH(\widehat{X};\mathbb{E}) \cong W^\nu H(\widehat{X};\mathbb{E})$.*

Here $\{m, n\}$ are the two middle perversities and $\{\mu, \nu\}$ are the two middle weight profiles.

We expect that the condition on the $\mathbb{Q}$-root system can be removed.

If $E^* \not\cong \overline{E}$, then $W^\eta H(\widehat{X};\mathbb{E}) = 0$ and $I_pH(\widehat{X};\mathbb{E})$ is a direct sum of weighted cohomology groups for smaller strata.

A brief survey of intersection cohomology

Let M be a smooth d-dimensional manifold and $\mathbb{E}$ a local coefficient system.

Ordinary cohomology $H(M;\mathbb{E})$ satisfies Poincaré duality:

$$H^i(M;\mathbb{E})^* \cong H_c^{d-i}(M;\mathbb{E}^*).$$

Poincaré duality fails if we replace M by a singular space Z. Goresky and MacPherson's intersection cohomology $I_pH(Z;\mathbb{E})$ agrees with ordinary cohomology if Z is smooth and continues to have Poincaré duality even if Z is not.

Data for Intersection Cohomology $I_pH(Z;\mathbb{E})$

(i) A d-dimensional stratified space Z:

- $Z = Z^0 = Z^1 \supseteq Z^2 \supseteq \cdots \supseteq Z^{d-1} \supseteq Z^d \supseteq Z^{d+1} = \emptyset$;
- the stratum $S^k := Z^k \setminus Z^{k+1}$ is a codimension-k manifold;
- $x \in S^k$ has a local neighborhood $U_x \cong B_{d-k} \times c(L_{k-1})$, where B_{d-k} is a ball and $c(L_{k-1})$ is a cone on a **link**.

(ii) A local coefficient system ($\equiv$ representation of π_1) $\mathbb{E}$ on S^0;

(iii) A **perversity** $p\colon \{2,\ldots,d\} \to \mathbb{Z}$,

$$p(2) = 0, \quad p(k) \leq p(k+1) \leq p(k) + 1.$$

Middle perversities: $n(k) = \left\lfloor \frac{(k-1)}{2} \right\rfloor$ and $m(k) = \left\lfloor \frac{(k-2)}{2} \right\rfloor$.

Simplicial Definition of Intersection Cohomology

Assume Z is piecewise-linearly stratified, $\mathcal{T}$ a triangulation.

$I_pC^j(Z,\mathbb{E})_{\mathcal{T}}$:= locally finite $(d-j)$-chains ξ (simplicial for $\mathcal{T}$) with values in $\mathbb{E}\otimes\mathcal{O}$, and satisfying the **allowability conditions** (for all $k\geq 2$):

$$\dim(|\xi|\cap Z_{d-k}) \leq d-j-k+p(k),$$
$$\dim(|\partial\xi|\cap Z_{d-k}) \leq d-j-1-k+p(k).$$

$I_pC^j(Z,\mathbb{E}) := \varinjlim I_pC^j(Z,\mathbb{E})_{\mathcal{T}}$.

The role is $p(k)$ is to bound the amount of intersection beyond the case of transversal intersection.

Note $\dim(|\xi|\cap Z_{d-2}) \leq d-j-2$ so that the top faces of ξ and $\partial\xi$ lie in S^0; thus "values in $\mathbb{E}\otimes\mathcal{O}$" makes sense.

$I_pC^j(Z,\mathbb{E})$ is a complex and its cohomology is $I_pH^j(Z,\mathbb{E})$

Key Properties of Intersection Cohomology $I_pH(Z;\mathbb{E})$

- $I_pH(Z;\mathbb{E})$ is stratification independent.

- Local calculation (determines $I_pH(Z;\mathbb{E})$ via Mayer-Vietoris):

$$I_pH^j(B_{d-k}\times c(L_{k-1});\mathbb{E}) \cong \begin{cases} I_pH^j(L_{k-1};\mathbb{E}) & \text{for } j \le p(k), \\ 0 & \text{for } j > p(k). \end{cases}$$

  For quotient of upper-half plane, a cusp has codimension 2 so one has local vanishing in degrees $j > p(2) = 0$. This $\Longrightarrow I_pH^1(X^*;\mathbb{E}) \cong H^1_P(X^*;\mathbb{E})$.

- (Poincaré duality) Let $q(k) = k - 2 - p(k)$ be the **dual perversity**. Then

$$I_pH^i(Z;\mathbb{E})^* \cong I_qH_c^{d-i}(Z;\mathbb{E}^*).$$

The Derived Category

More useful to consider intersection cohomology as hypercohomology of object $\mathcal{I}_p\mathcal{C}(Z;\mathbb{E})$ in derived category $\mathrm{D}^b(Z)$. Good references: Borel, *et al.* (1984), Kashiwara and Schapira (1990).

Objects of $\mathrm{D}^b(Z)$: bounded complexes of sheaves

$$\mathcal{S} \equiv \qquad \cdots \xrightarrow{d_{n-2}} \mathcal{S}^{n-1} \xrightarrow{d_{n-1}} \mathcal{S}^{n} \xrightarrow{d_n} \cdots$$

Before defining morphisms recall cohomology sheaf:

$$H(\mathcal{S}) \equiv \qquad \cdots \xrightarrow{0} \mathrm{Ker}\, d_{n-1}/\,\mathrm{Im}\, d_{n-2} \xrightarrow{0} \mathrm{Ker}\, d_n/\,\mathrm{Im}\, d_{n-1} \xrightarrow{0} \cdots$$

A morphism of complexes $\psi\colon \mathcal{S} \to \mathcal{S}'$ induces $H(\psi)\colon H(\mathcal{S}) \to H(\mathcal{S}')$; ψ is a **quasi-isomorphism** if $H(\psi)$ is an isomorphism.

Morphisms of $\mathrm{D}^b(Z)$: certain equivalence class of diagrams $\mathcal{S} \xleftarrow{\phi_1} \mathcal{S}'' \xrightarrow{\phi_2} \mathcal{S}'$ where ϕ_1 is a quasi-isomorphism.

Quasi-isomorphism of complexes of sheaves becomes isomorphism in derived category.

Derived category can be viewed as "localization" of homotopy category of complexes of sheaves by "inverting" quasi-isomorphisms.

Constructible derived category $\mathrm{D}^b_c(Z)$: full subcategory with objects $\mathcal{S}$ such that $H(\mathcal{S})|_{S^k}$ is a locally constant sheaf with finitely generated stalks for all strata S^k.

Functors on the Derived Category

Every complex of sheaves $\mathcal{S}$ has an **injective resolution**, $\mathcal{S} \to \mathcal{I}$, where $\mathcal{I}$ is a complex of injective sheaves.

Given a left exact functor F on sheaves, define the **right derived functor** RF on $\mathbf{D}^b(Z)$: $RF(\mathcal{S}) := F(\mathcal{I})$.

Examples:

- For $k\colon Z \to W$ continuous we have functor Rk_*:
$$Rk_*\mathcal{S}(V) := \mathcal{I}(k^{-1}(V)) \text{ for } V \subseteq W \text{ open.}$$
- k^* is exact so $Rk^*\mathcal{S}(U) = k^*\mathcal{S}(U) = \mathcal{S}(k(U))$ for $U \subseteq Z$ open.
- **Hypercohomology**: $H(Z;\mathcal{S}) := H(R\Gamma(Z;\mathcal{S})) = H(\Gamma(Z;\mathcal{I}))$.

Often one can replace $\mathcal{I}$ by a resolution by fine or soft sheaves.

Rk_* is a right adjoint of k^*:

$$\mathsf{Mor}_{\mathbf{D}^b(Z)}(k^*\mathcal{S}, \mathcal{S}') = \mathsf{Mor}_{\mathbf{D}^b(W)}(\mathcal{S}, Rk_*\mathcal{S}').$$

Apply this with $\mathcal{S}' = k^*\mathcal{S}$ and the identity morphism to obtain the **adjunction morphism**: $\mathcal{S} \longrightarrow Rk_*k^*\mathcal{S}$.

There is also a functor $k^!$ which is a right adjoint to $k_!$. In case k is the inclusion of a locally closed subset, $k^! = k^* \circ R\Gamma_Z$, the derived "sections supported by Z" functor:

$$k^!\mathcal{S}(U) \equiv \mathsf{Ker}\Big(\Gamma(\widetilde{U}; \mathcal{I}) \to \Gamma(\widetilde{U} \setminus (Z \cap \widetilde{U}); \mathcal{I})\Big),$$

where $\widetilde{U} \subseteq W$ is a open such that $U = Z \cap \widetilde{U}$ is closed in $\widetilde{U}$.

Sheaf Theoretic Intersection Cohomology

Define the truncation functor $\tau^{\leqslant n}$ by

$$\tau^{\leqslant n}\mathcal{S} := \quad \cdots \xrightarrow{d_{n-2}} \mathcal{S}^{n-1} \xrightarrow{d_{n-1}} \ker d_n \xrightarrow{d_n} 0 \longrightarrow \cdots \quad .$$

For all $k \geq 2$, define

$$\begin{aligned} &U_k := Z \setminus Z_{d-k}, \\ &j_k \colon U_k \hookrightarrow U_{k+1} = U_k \cup S^k \text{ and} \\ &i_k \colon S^k \hookrightarrow U_{k+1} \text{ the inclusions.} \end{aligned}$$

Encode the local calculation into an object of $\mathbf{D}_c^b(Z)$ by Deligne's formula:

$$\mathcal{I}_p\mathcal{C}(Z;\mathbb{E}) := \tau^{\leqslant p(d)} Rj_{d*} \cdots \tau^{\leqslant p(3)} Rj_{3*} \tau^{\leqslant p(2)} Rj_{2*}\mathbb{E},$$

Intersection cohomology $I_pH(Z;\mathbb{E})$ is defined as the hypercohomology of $\mathcal{I}_p\mathcal{C}(Z;\mathbb{E})$.

Theorem (Goresky and MacPherson (1983)). *Assume that* $\mathcal{S}$ *is an object of* $\mathbf{D}^b_c(Z)$ *satisfying*:

(i) $Ri_2^*\mathcal{S} \cong \mathbb{E}$.

(ii) $H^j(i_k^*\mathcal{S}) = 0$ *for* $j > p(k)$.

(iii) *The* **attaching morphism** $i_k^*\mathcal{S} \to i_k^* R j_{k*} j_k^* \mathcal{S}$ *induces an isomorphism on cohomology sheaves in degrees* $\leq p(k)$.

Then there is a natural isomorphism $\mathcal{S} \cong \mathcal{I}_p\mathcal{C}(Z;\mathbb{E})$.

• One can also give a stratification-independent version of the theorem. Thus intersection cohomology is a topological invariant.

• The sheaf $U \mapsto I_pC(U;\mathbb{E})$ of allowable chains is soft and satisfies the above conditions. Thus the sheaf-theoretic definition of $I_pH(Z;\mathbb{E})$ agrees with the simplicial definition.

Sheaf-theoretic L^2-cohomology

Assume the nonsingular stratum S^0 has a Riemannian metric and $\mathbb{E}$ has a fiber-wise metric.

For $W \subseteq S^0$ open, let $A_{(2)}(W;\mathbb{E}) :=$ smooth $\mathbb{E}$-valued forms ϕ on W such $\int_W |\phi|^2\, dV < \infty$ and $\int_W |d\phi|^2\, dV < \infty$.

Define the L^2-**cohomology sheaf** $\mathcal{L}_{(2)}(Z;\mathbb{E})$ via the presheaf

$$U \mapsto A_{(2)}(U \cap S^0;\mathbb{E})$$

When $\mathcal{L}_{(2)}(Z;\mathbb{E})$ is fine, $H(Z;\mathcal{L}_{(2)}(Z;\mathbb{E})) = H_{(2)}(S^0;\mathbb{E})$.

Zucker's conjecture is proved by verifying $\mathcal{L}_{(2)}(X^*;\mathbb{E})$ is fine (which is easy) and satisfies the local vanishing condition of middle-perversity intersection cohomology (which is hard).