COHOMOLOGY OF LOCALLY SYMMETRIC SPACES AND THE MODULI SPACE OF CURVES

LESLIE SAPER

Let Γ be a group and E a Γ-module. We are interested in the cohomology $H^{\cdot}(\Gamma; E)$. If X is a contractible space on which Γ acts properly one may represent this cohomology topologically as $H^{\cdot}(\Gamma\backslash X; \mathbb{E})$ for a certain sheaf $\mathbb{E}$. Our primary interest is when Γ is arithmetic and X is a symmetric space or when Γ is a mapping class group and X is Teichmüller space. When $M = \Gamma\backslash$ is a compact Riemannian manifold it is profitable to represent cohomology by harmonic forms from which one can prove Poincaré duality. When M is non-compact the same can essentially be done but for the L^2-cohomology $H^{\cdot}_{(2)}(M; \mathbb{E})$, which is an invariant of the quasi-isometry class of the metric.

We consider three examples to indicate that $H^{\cdot}_{(2)}(M; \mathbb{E})$ can represent a topological invariant: (1) Cheeger's horn metrics on triangulated pseudomanifolds; (2) Saper's proof that the L^2-cohomology of the Weil-Petersson metric on the moduli space of curves $\mathcal{M}_g$ is the cohomology of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$,; and (3) Zucker's conjecture (proved by Saper-Stern and Looijenga) that the L^2-cohomology of a Hermitian locally symmetric space $\Gamma\backslash X$ is the middle perversity intersection cohomology of the Baily-Borel Satake compactification $\Gamma\backslash X^*$. We conclude this section with a heuristic for Zucker's conjecture.

Example (2) above answered a question of Hain and Looijenga, perhaps motivated in analogy with Zucker's conjecture. A better analogy suggests one consider the Siegel metric on $\mathcal{M}_g$, the pull-back of the locally symmetric metric under the Torelli embedding $\tau\colon \mathcal{M}_g \to \mathcal{A}_g$, and the Satake compactification $\mathcal{M}_g^*$, the closure of $\tau(\mathcal{M}_g)$ in the Baily-Borel Satake compactification $\mathcal{A}_g^*$. We conjectured in 1993 that the analogue of Zucker's conjecture holds in this setting. Although no progress has been made on this conjecture for $g > 3$, more recent work on Rapoport's conjecture suggests a possible approach.

Rapoport's conjecture (made independently by Goresky and MacPherson) asserts that for a Hermitian symmetric space X, either middle perversity intersection cohomology of the reductive Borel-Serre compactification $\Gamma\backslash\overline{X}^{RBS}$ is isomorphic to the middle perversity intersection cohomology of $\Gamma\backslash X^*$. The conjecture was motivated by Langlands's program—the point is that $\Gamma\backslash\overline{X}^{RBS}$ is a far less singular compactification making local calculations easier. Saper proved the conjecture (actually a generalization to equal-rank spaces) in 2001 using $\mathcal{L}$-modules, a combinatorial model of sheaves on $\Gamma\backslash\overline{X}^{RBS}$.

In current work $\mathcal{L}$-modules are being used to study $H^{\cdot}(\Gamma; E)$ itself for Γ arithmetic. We now suggest that an analogue of $\mathcal{L}$-modules can be applied to address the 1993 conjecture on the moduli space of curves. Namely $\overline{\mathcal{M}_g}$ could play the role of the reductive Borel-Serre compactification. What is needed is to understand the

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Siegel metric locally on $\overline{\mathcal{M}_g}$ (as opposed to locally on $\mathcal{M}_g^*$), understand the fibers of the extended Torelli map $\overline{\mathcal{M}_g} \to \mathcal{M}_g^*$, and prove a vanishing theorem on these fibers. Progress in some simple concrete examples has been made.

Department of Mathematics, Duke University, Box 90320, Durham, NC 27708, U.S.A.
E-mail address: saper@math.duke.edu
URL: http://www.math.duke.edu/faculty/saper