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Cohomology of Locally Symmetric Spaces

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Data:

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$D = G/K$, a symmetric space of noncompact type,

$\Gamma \subset G$, an arithmetically defined torsion-free discrete subgroup,

$X = \Gamma \backslash D$, the corresponding locally symmetric space,

E, a finite dimensional representation of G, and

$\mathbb{E} = D \times_{\Gamma} E$, the corresponding locally constant sheaf on X.

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We will address these questions via a specific problem . . .

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Problem:

Give a topological description of $\mathcal{H}_{(2)}(X;\mathbb{E})$.

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A solution is given by the Hodge-de Rham isomorphism:

$$\mathcal{H}_{(2)}(X;\mathbb{E}) \cong H_{\mathsf{dR}}(X;\mathbb{E}) \cong H(X;\mathbb{E})$$

where

$$H_{\mathsf{dR}}(X;\mathbb{E}) = \frac{\operatorname{Ker} d}{\operatorname{Im} d} = \frac{\text{closed forms}}{\text{exact forms}} = \text{de Rham cohomology,}$$

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Sketch of proof:

- All smooth forms are L^2.
- $\mathrm{Im}\, d$ is closed so $\dfrac{\mathrm{Ker}\, d}{\mathrm{Im}\, d} = (\mathrm{Ker}\, d) \cap (\mathrm{Im}\, d)^{\perp} = \mathrm{Ker}\, \Delta$.
- Poincaré lemma.

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But the Hodge-de Rham isomorphism fails in the noncompact case in general.

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Theorem. *If* $\dim H_{(2)}(X;\mathbb{E}) < \infty$, *then* $H_{(2)}(X;\mathbb{E}) \cong \mathcal{H}_{(2)}(X;\mathbb{E})$ *and satisfies Poincaré duality.*

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Proof:

$$\operatorname{Ker} d \,/\operatorname{Im} d = \left(\operatorname{Ker} d \Big/ \overline{\operatorname{Im} d}\right) \bigoplus \left(\overline{\operatorname{Im} d} \Big/ \operatorname{Im} d\right)$$

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D &= H = \mathsf{SL}_2(\mathbb{R}) / \mathsf{SO}(2) \\
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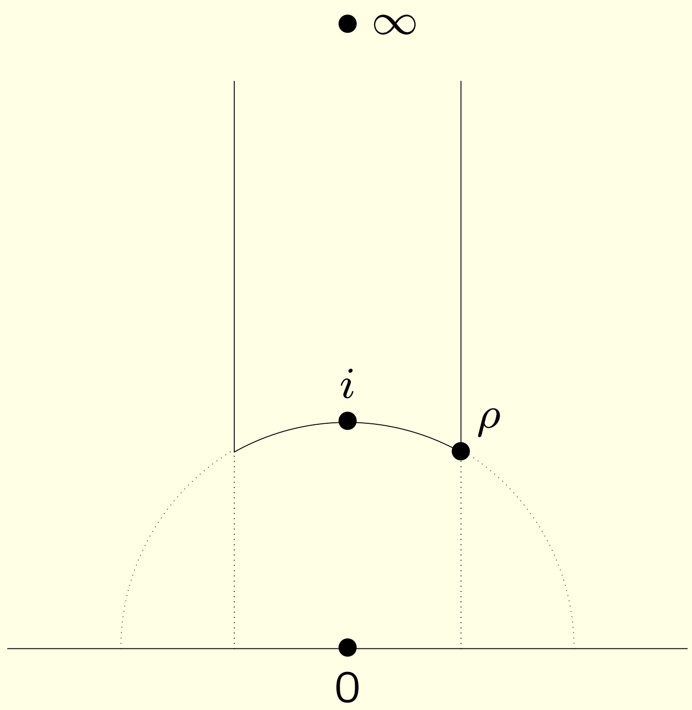

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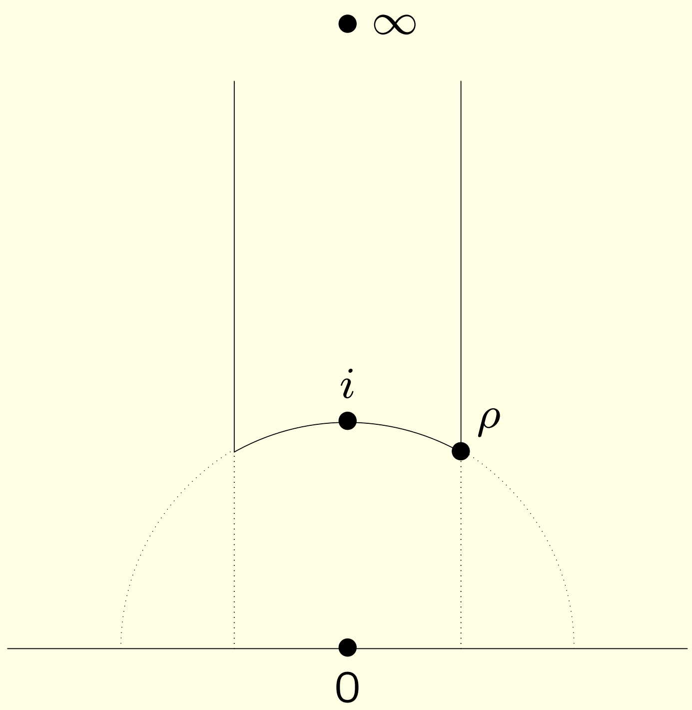

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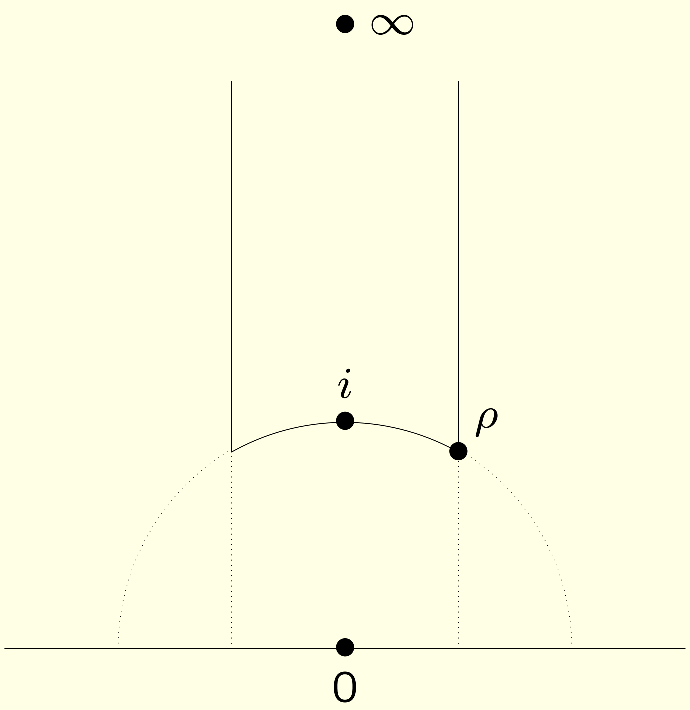

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$\mathcal{S}_{k+2}(\Gamma)$ = classical holomorphic modular cusp forms of weight $k+2$, that is, $f\colon H \to \mathbb{C}$ holomorphic,

$$
f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,
$$

and f vanishes at all cusps.

In this example the L^2-harmonic 1-forms are known:

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But we want a *topological* interpretation:

$H^* = H \cup \mathbb{Q} \cup \{\infty\}$,

$X^* = \Gamma\backslash H^* = X \cup \{\text{cusps}\}$, a projective algebraic curve defined over a number field,

$H^1_P(X^*;\mathbb{E}) = \mathrm{Ker}\Big(H^1(X;\mathbb{E}) \to \bigoplus_{p\in\{\text{cusps}\}} H^1(U_p;\mathbb{E})\Big)$ (*parabolic cohomology*).

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Theorem (Eichler-Shimura). $\mathcal{S}_{k+2}(\Gamma) \oplus \overline{\mathcal{S}_{k+2}(\Gamma)} \cong H^1_P(X^*;\mathbb{E})$.

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- Thus L^2-cohomology equals parabolic cohomology.

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Question: What replaces X^* and H^1_P?

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Baily and Borel show that X^* is a (generally singular) projective algebraic variety.

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The *intersection cohomology* $I_pH(X^*;\mathbb{E})$ is characterized by the local calculation:

$$I_pH^i(\mathrm{Ball}_{d-k} \times \mathrm{cone}(L_{k-1});\mathbb{E}) \cong \begin{cases} I_pH^i(L_{k-1};\mathbb{E}) & \text{for } i \le p(k), \\ 0 & \text{for } i > p(k) \end{cases}$$

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Here we always take $p(k)$ to be a *middle perversity*:

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Intersection cohomology was introduced by Goresky and MacPherson in order to restore Poincaré duality to the cohomology of singular spaces.

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In view of the fact that X^* is naturally defined over a number field, this result is important for Langlands's program.

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For Y smooth, one obtains the *Hasse-Weil zeta function* of Y, which encodes $\#Y(\mathbb{F}_{p^k})$ for all prime powers p^k.

Thus Langlands's program predicts

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- $Y =$ an elliptic curve: the Shimura-Taniyama-Weil conjecture ($\Longrightarrow$ Fermat's last theorem);
- $Y = X^*$: our case.

Compare L-functions via fixed-point formulas:

$$\begin{array}{c} \text{Lefschetz fixed} \\ \text{point formula for} \\ \text{Frobenius on } I_pH(X^*;\mathbb{E}) \end{array} \overset{?}{\longleftrightarrow} \begin{array}{c} \text{Arthur-Selberg trace} \\ \text{formula for Hecke} \\ \text{operators on } H_{(2)}(X;\mathbb{E}) \end{array}$$

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- Local contributions on left involve local intersection cohomology of X^* — hard since the links are complicated.
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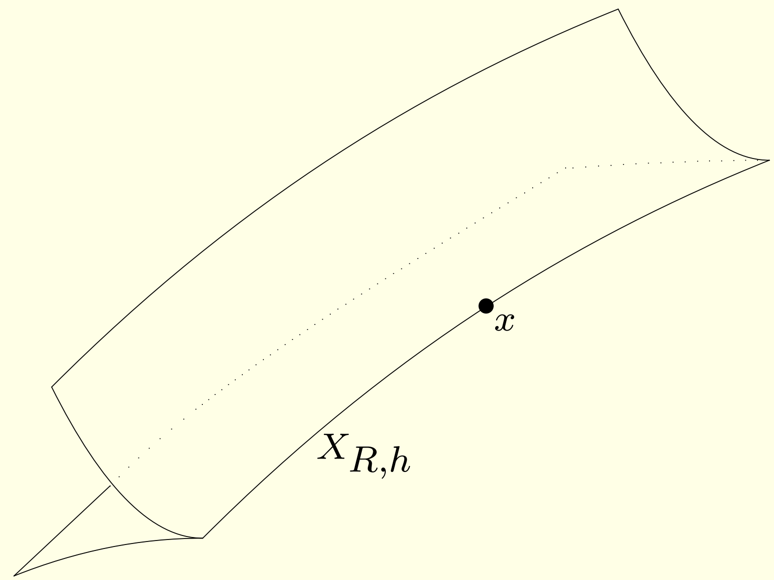
- Many other difficulties.

Structure of X^* near a stratum $X_{R,h}$:



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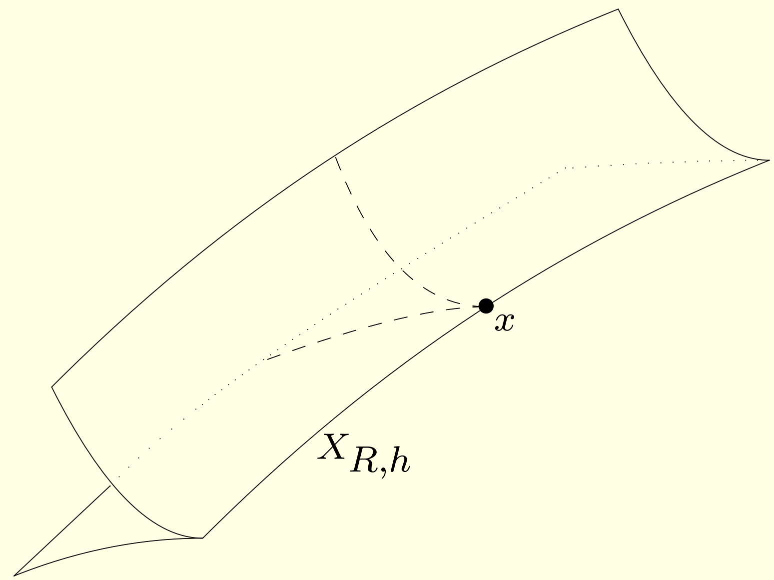

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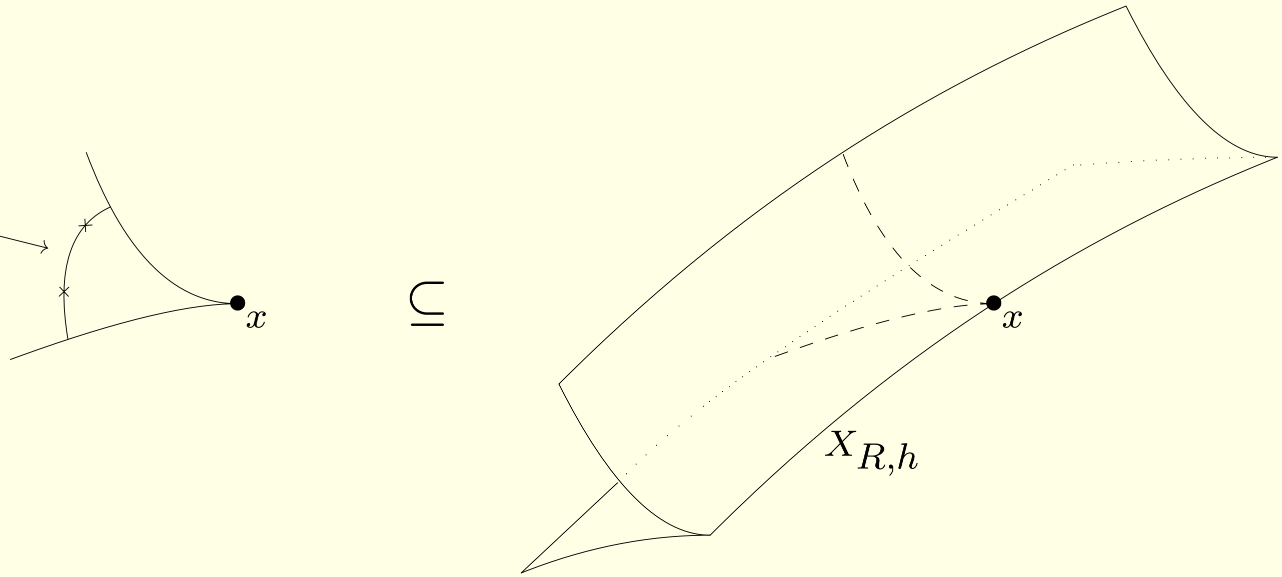
Link$^\circ$ =
$\mathcal{N}_R$-bundle
over $X_{R,\ell}$



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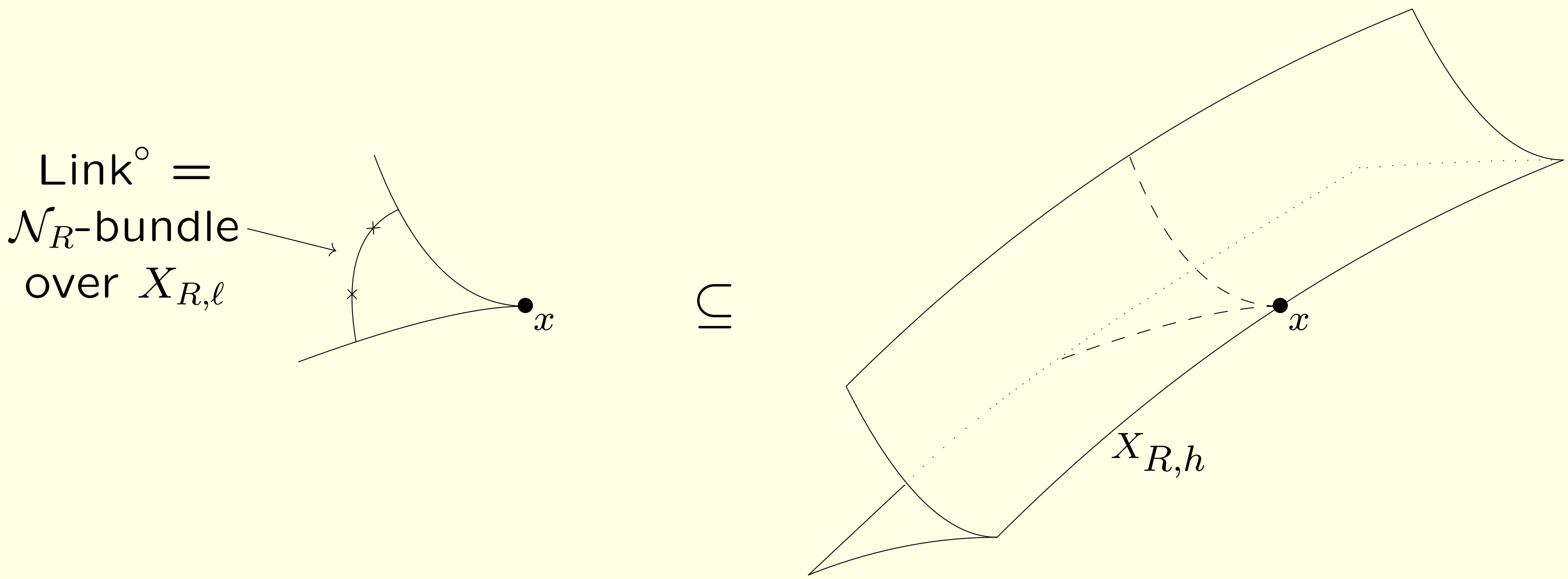

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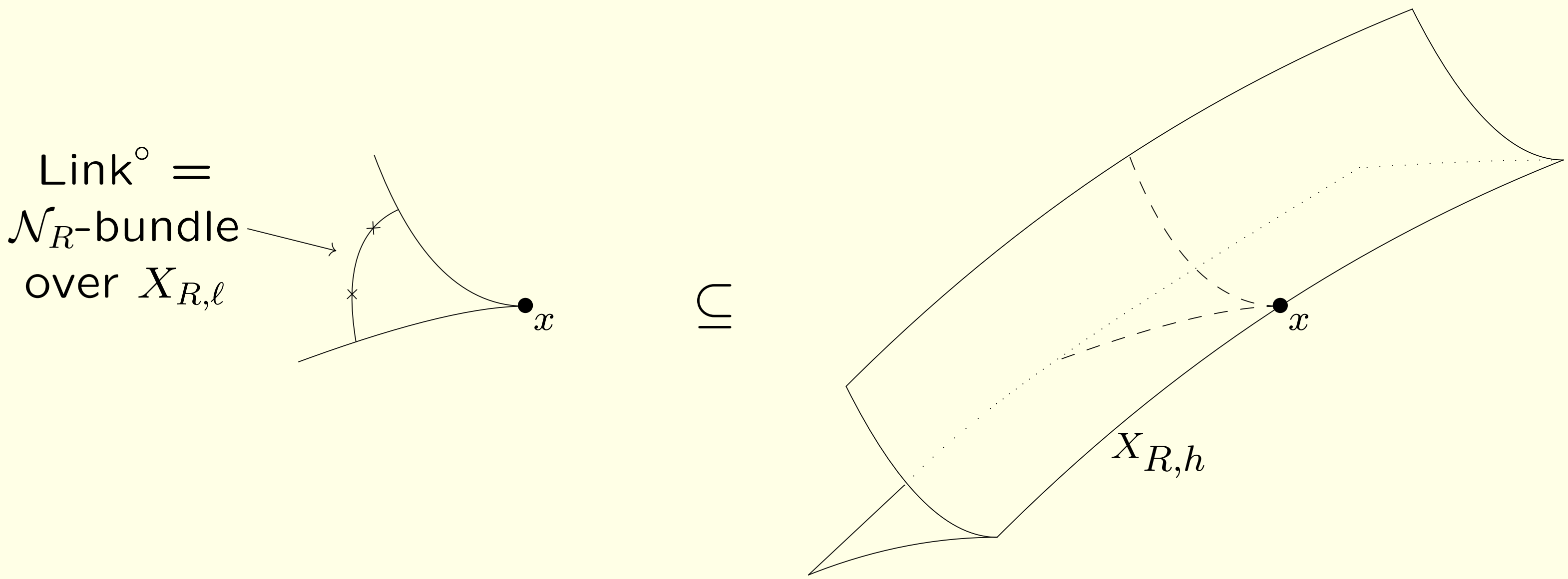

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Since we do not have an effective method in general to compute the cohomology of a locally symmetric space, the local intersection cohomology of X^* is difficult to work with.

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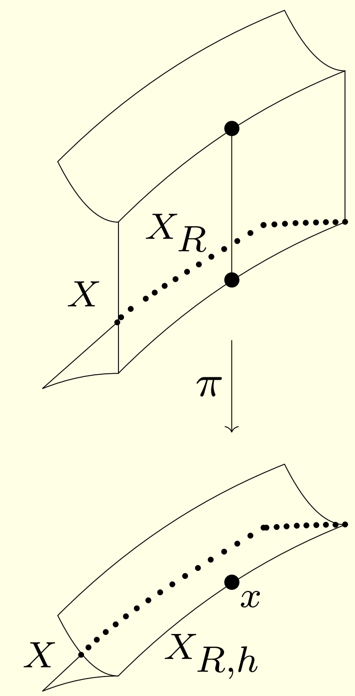
Theorem (S.). *The conjecture above is true.*

The reductive Borel-Serre compactification $\widehat{X}$

Three constructions:



(i) "Blow up" each stratum of X^* (replace each point with its link) and collapse the nilmanifold fibers

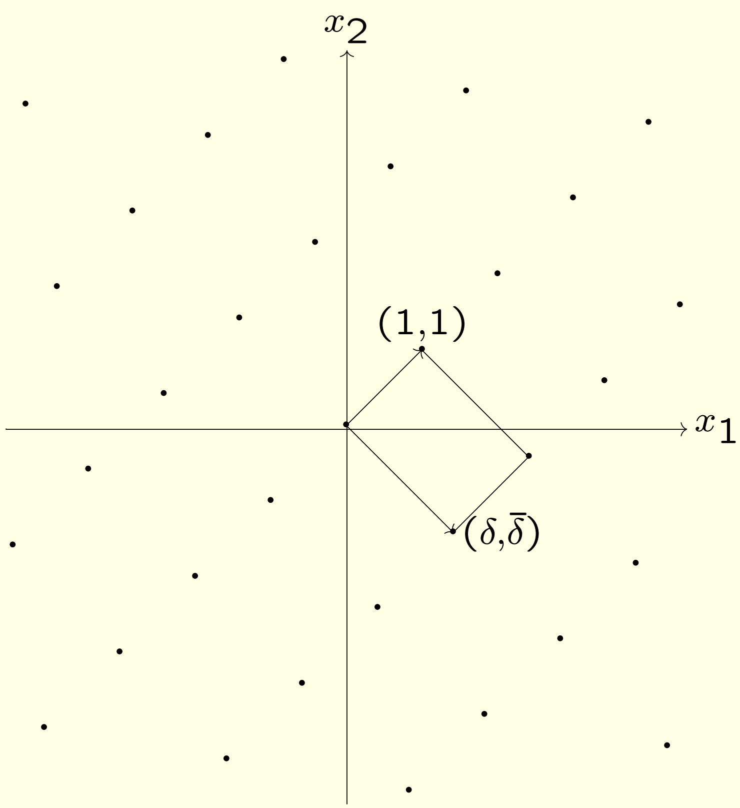
(ii) Remove a neighborhood of each stratum of X^* and collapse the nilmanifold fibers on boundary faces

(iii) Start with the Borel-Serre compactification (1973) $\overline{X}$ (a manifold with corners) and collapse the nilmanifold fibers on the boundary faces (applies to any locally symmetric space X)

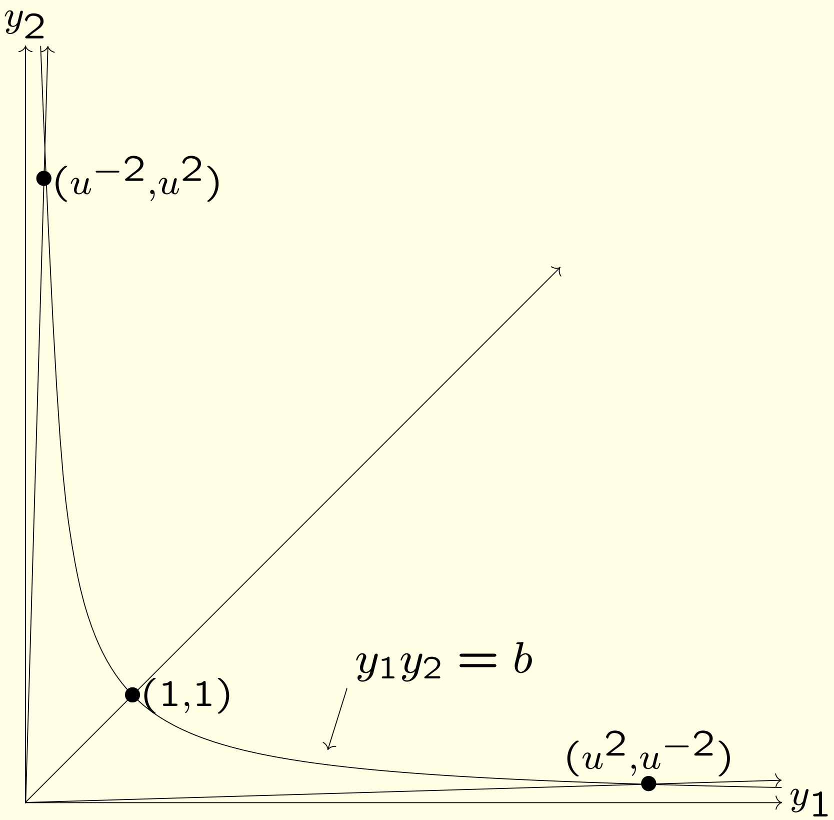
Example: Hilbert Modular Surface $\mathrm{SL}(2,\mathcal{O}_k)\backslash(H\times H)$

Here $k=\mathbb{Q}(\sqrt{d})$, $d>0$. Near "infinity", $\mathrm{SL}(2,\mathcal{O}_k)$ acts via

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a\in\mathcal{O}_k \right\} \quad \rtimes \quad \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \middle| \, u\in\mathcal{O}_k^{\times} \right\}$$



$\mathcal{O}_k=\mathbb{Z}+\mathbb{Z}\delta$



$\mathcal{O}_k^{\times}=\{\,u^k \mid k\in\mathbb{Z}\,\}$

Thus

	Boundary stratum	Link
$\overline{X}$	flat T^2-bundle over S^1	point
$\widehat{X}$	S^1	T^2
X^*	point	flat T^2-bundle over S^1

The hyperbola $y_1 y_2 = b$ in the $y_1 y_2$-plane becomes the S^1 above under the action of $\left\{ \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix} \right) \middle| \, u \in \mathcal{O}_k^\times \right\}$. The T^2-fibers correspond to the $x_1 x_2$-plane modulo a lattice.

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In general, the space S^1 above will be replaced by a locally symmetric space X_P; the fibers T^2 above will be replaced in general by a compact nilmanifold $\mathcal{N}_P$. Here P is a Γ-conjugacy class of parabolic $\mathbb{Q}$-subgroups of G; these index the strata.

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The theory of $\mathcal{L}$-modules and micro-support . . .

The poset $\mathcal{P}$

$\mathcal{P} = \Gamma$-conjugacy classes of parabolic $\mathbb{Q}$-subgroups. For example (when $\mathbb{Q}$-rank $G = 2$):

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
Q_1 \arrow[dr, dash] & & Q_2 \arrow[dl, dash] \\
 & P &
\end{tikzcd}$$

The Levi quotients $\mathcal{L} = \mathcal{L}_{\mathcal{P}}$

Pass to the reductive Levi quotients $L_Q = Q/N_Q$ for all $Q \in \mathcal{P}$.

$$\begin{array}{ccc} & L_G & \\ L_{Q_1} & & L_{Q_2} \\ & L_P & \end{array}$$

An $\mathcal{L}$-module $\mathcal{M}$

An $\mathcal{L}$-module consists of graded L_Q-modules E_Q for all Q . . .

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 & E_G & \\
E_{Q_1} & & E_{Q_2} \\
 & E_P &
\end{tikzcd}$$

An $\mathcal{L}$-module $\mathcal{M}$

and degree 1 morphisms $f_{PQ} : H(\mathfrak{n}_P^Q; E_Q) \xrightarrow{[1]} E_P$ for all $P \leq Q$

$$
\begin{tikzcd}
 & & E_G & & \\
 & H(\mathfrak{n}_{Q_1}, E_G) \arrow[dl, "f_{Q_1G}"', "{[1]}"] & & H(\mathfrak{n}_{Q_2}, E_G) \arrow[dr, "f_{Q_2G}", "{[1]}"'] & \\
E_{Q_1} & & H(\mathfrak{n}_P, E_G) \arrow[dd, "{[1]}"', "f_{PG}"] & & E_{Q_2} \\
 & H(\mathfrak{n}_P^{Q_1}, E_{Q_1}) \arrow[dr, "f_{PQ_1}", "{[1]}"'] & & H(\mathfrak{n}_P^{Q_2}, E_{Q_2}) \arrow[dl, "f_{PQ_2}"', "{[1]}"] & \\
 & & E_P & &
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An $\mathcal{L}$-module $\mathcal{M}$

satisfying $\sum_{P \leq Q \leq R} f_{PQ} \circ H(\mathfrak{n}_P^Q; f_{QR}) = 0$ for all $P \leq R$.

$$
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& \mathcal{A}_{\mathsf{sp}}(\widehat{X}_P;\mathbb{E}_P) &
\end{tikzcd}
$$

The realization $\mathcal{S}_{\widehat{X}}(\mathcal{M})$ with $d_{\cdot\cdot}$ factored

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{A}_{\mathsf{sp}}(\widehat{X};\mathbb{E}_G) \arrow[dl, "k_{Q_1G}"'] \arrow[dd, "k_{PG}"] \arrow[dr, "k_{Q_2G}"] & & \\
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\end{tikzcd}
$$

The micro-support $\mathrm{SS}(\mathcal{M})$ of an $\mathcal{L}$-module $\mathcal{M}$

Roughly $\mathrm{SS}(\mathcal{M})$ consists of all irreducible representations V of L_P (any $P \in \mathcal{P}$) such that

$$(V|_{M_P})^* \cong \overline{V|_{M_P}}, \text{ and}$$

$$H(i_P^* \widehat{\imath}^!_{Q_V} \mathcal{M})_V = H(U, U \setminus (U \cap \widehat{X}_{Q_V}); \mathcal{M})_V \neq 0.$$

Here we write $L_P = M_P A_P$ where A_P is the $\mathbb{Q}$-split center of L_P and $Q_V \geq P$ is chosen depending on the character by which A_P acts on V. Finally U is a small neighborhood of a point on the P-stratum X_P.

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- To every $\mathcal{L}$-module $\mathcal{M}$ there is associated an invariant called the *micro-support* $\mathrm{SS}(\mathcal{M})$, it is a finite collection of irreducible finite-dimensional representations of all Levi quotients L_P.

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- Our various cohomological invariants equal $H(\widehat{X};\mathcal{M}) := H(\widehat{X};\mathcal{S}(\mathcal{M}))$ for various $\mathcal{L}$-modules $\mathcal{M}$;

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Our results follow from three theorems on $\mathcal{L}$-modules:

- a **Vanishing Theorem** for global cohomology;

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- a **Functoriality Theorem** for micro-support.

Vanishing Theorem for the Cohomology of an $\mathcal{L}$-module

Define

$$c(\mathcal{M}) = \inf_{V \in \mathsf{SS}(\mathcal{M})} \tfrac{1}{2}(\dim D_P - \dim D_P(V)) + c(V;\mathcal{M}) \ ,$$
$$d(\mathcal{M}) = \sup_{V \in \mathsf{SS}(\mathcal{M})} \tfrac{1}{2}(\dim D_P + \dim D_P(V)) + d(V;\mathcal{M}) \ .$$

The first terms are the range of degrees where $H_{(2)}(X_P;\mathbb{V})$ can be nonzero by a vanishing theorem of Raghunathan.

The second terms are computed combinatorially from the micro-support.

Vanishing Theorem. $H^i(\widehat{X};\mathcal{M}) = 0$ *for* $i \notin [c(\mathcal{M}), d(\mathcal{M})]$.

In particular, $H(\widehat{X};\mathcal{M}) \equiv 0$ if $\mathsf{SS}(\mathcal{M}) = \emptyset$.

Micro-support of Intersection Cohomology

Micro-support is not always so easy to compute. The following is a very deep combinatorial result.

Micro-Purity Theorem. *Assume the $\mathbb{Q}$-root system of G does not contain a factor of type D_n, E_n, or F_4. Let p be a middle perversity. If $E^* \cong \overline{E}$, then* $\mathsf{SS}(\mathcal{I}_p\mathcal{C}(\widehat{X}; E)) = \{E\}$.

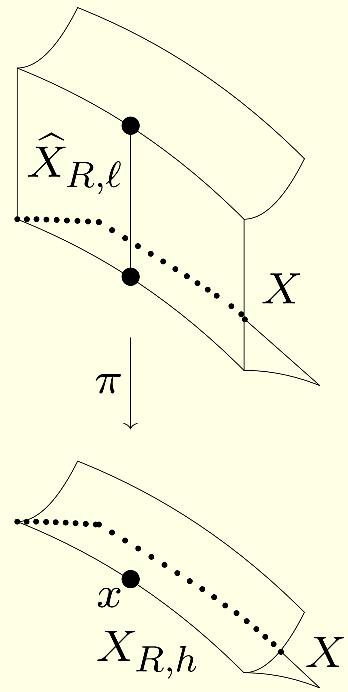
A simpler result is

Theorem. *If $E^* \cong \overline{E}$, then* $\mathsf{SS}(\mathcal{L}_{(2)}(\widehat{X}; E)) = \{E\}$.

Functoriality of Micro-support

Let $\mathcal{M}$ be an $\mathcal{L}$-module for which $\mathrm{SS}(\mathcal{M}) = \{E\}$ (e.g. $\mathcal{I}_p\mathcal{C}(\widehat{X}; E)$ or $\mathcal{L}_{(2)}(\widehat{X}; E)$).

Let $\pi : \widehat{X} \to X^*$ be the projection onto a Satake compactification with equal-rank real boundary components.



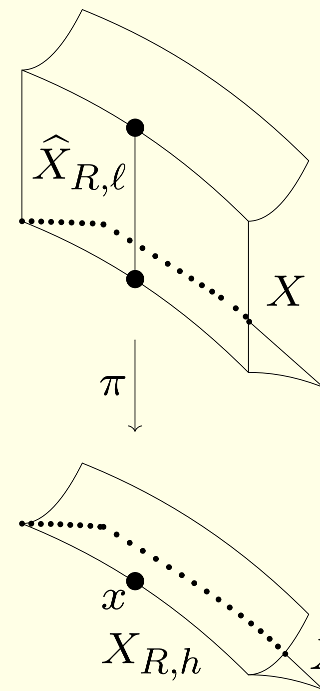
To prove Zucker and Rapoport's conjecture, we need to check the local vanishing condition for the pushforward of $\mathcal{M}$ by π. Equivalently we need to show

$$H^i(\pi^{-1}(x); \mathcal{M}|_{\pi^{-1}(x)}) = 0 \qquad \text{for } i > \tfrac{1}{2}\operatorname{codim} X_{R,h} - 1.$$

However $\pi^{-1}(x) \cong \widehat{X}_{R,\ell} \times \{x\}$.

The Vanishing Theorem implies

$$H^i(\widehat{X}_{R,\ell}; \mathcal{M}|_{\widehat{X}_{R,\ell}}) = 0 \quad \text{for } i > d(\mathcal{M}|_{\widehat{X}_{R,\ell}}).$$



Thus the following theorem completes the proof:

Functoriality Theorem. *Let $\mathcal{M}$ be an $\mathcal{L}$-module with* $\mathsf{SS}(\mathcal{M}) = \{E\}$ *and let $X_{R,h}$ be a stratum of a Satake compactification X^* with real equal-rank boundary components. Then*

$$d(\mathcal{M}|_{\widehat{X}_{R,\ell}}) \leq \tfrac{1}{2} \operatorname{codim} X_{R,h} - 1 \ .$$

Final remark:

$\mathcal{L}$-modules have many other applications besides the Rapoport-Goresky-MacPherson conjecture. For example:

Theorem (S., Li-Schwermer). *If E has regular highest weight, then*

$$H^i(X;E) = 0 \qquad \text{for } i < \frac{1}{2}\Big(\dim X - (\operatorname{rank} G - \operatorname{rank} K)\Big).$$