Math 431 February 21, 2018

Office Problem

(Office consultation: First Draft due March 2)

Let $\mathcal{C}(\mathbb{Q})$ denote the set of equivalence classes $[a_n]$ of Cauchy sequences $\{a_n\}$ of rational numbers as defined in Assignment 5. The purpose of this office problem is to complete the construction of the real numbers from the set $\mathcal{C}(\mathbb{Q})$ which was begun in Assignment 5. In particular, you will show that $\mathcal{C}(\mathbb{Q})$ is ordered and Archimedean, define completeness in $\mathcal{C}(\mathbb{Q})$ and show that each"infinite decimal" gives rise to an element of $\mathcal{C}(\mathbb{Q})$.

1. A Cauchy sequence $\{a_n\}$ is called *positive* if there exists $B \in \mathbb{Q}$, $B > 0$, and $N \in \mathbb{N}$ such that if $n \geq N$ then $a_n \geq B$. Prove that if $\{a_n\} \sim \{b_n\}$ and $\{a_n\}$ is positive, then $\{b_n\}$ is also positive.

Now define $P \subseteq \mathcal{C}(\mathbb{Q})$ to be the subset consisting of those $[a_n]$ such that $\{a_n\}$ is positive. By 1. this notion is well-defined, *i.e.* does not depend on the choice of Cauchy sequence in the equivalence class $[a_n]$.

2. Prove that if $[a_n], [b_n] \in P$, then $[a_n] +_{\mathcal{C}(\mathbb{Q})} [b_n] \in P$ and $[a_n] \cdot_{\mathcal{C}(\mathbb{Q})} [b_n] \in P$.

3. Prove that if $[a_n] \in \mathcal{C}(\mathbb{Q})$, then either $[a_n] \in P$, $[a_n] = 0$ or $-[a_n] \in P$; and that only one of these conditions holds.

4. Prove that $\mathcal{C}(\mathbb{Q})$ satisfies the Archimedean property.

The Additional Problems in Assignment 5 essentially proved that $\mathcal{C}(\mathbb{Q})$ is a field. Hence we see from 2., 3. and 4. that $\mathcal{C}(\mathbb{Q})$ is an *Archimedean ordered* field. In fact, $\mathcal{C}(\mathbb{Q})$ has the four equivalent properties: Axiom C, l.u.b. Axiom, Cauchy sequences converge plus the Archimedean Principle, and BW. But *these are not axioms in* $\mathcal{C}(\mathbb{Q})$: they are *provable from the definition* of $\mathcal{C}(\mathbb{Q})$ (and once you know one of them holds, the others do as well, since the proofs of their equivalence use only the properties of Archimedean ordered fields). This is a little harder to do, and the next problem asks you just to state what would have to be proved.

5. Give a clear statement of the assertion that every Cauchy sequence in $\mathcal{C}(\mathbb{Q})$ converges to an element of $\mathcal{C}(\mathbb{Q})$. (To do this, you must first define "Cauchy sequence in $\mathcal{C}(\mathbb{Q})$".)

The final problem shows the connection between the usual concept of real number and Cauchy sequences: "decimals" do give us elements of $\mathcal{C}(\mathbb{Q})$.

6. Let $\{d_n\}$ be a sequence where $d_i \in \{0, 1, \ldots, 9\}$ for all i. Show that the sequence $\{s_n\}$ where

$$s_n = \sum_{k=1}^{k=n} d_k 10^{-k}$$

is Cauchy. (We thus think of $[s_n]$ as representing the decimal $.d_1 d_2 d_3 \ldots$)

So $\mathcal{C}(\mathbb{Q})$ talks like $\mathbb{R}$, walks like $\mathbb{R}$,...