Math 431 February 16, 2018

Assignment 5
(Due February 23)

Reading: (*from Reed*) §6.1, 3.2

Problems: §2.4: #10
§2.6: #1, 3, 9
§6.1: #1(a,c), 8

Additional Problems: 1. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences in an ordered field F. Let $\{a_n\} \sim \{b_n\}$ mean that $a_n - b_n \to 0$. Prove that $\sim$ is an equivalence relation: $\{a_n\} \sim \{a_n\}$; if $\{a_n\} \sim \{b_n\}$ then $\{b_n\} \sim \{a_n\}$; if $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$, then $\{a_n\} \sim \{c_n\}$.

2. Let $\mathcal{C}(F)$ denote the set of equivalence classes of Cauchy sequences in F. Find an injective function $F \to \mathcal{C}(F)$. (So we can think of F as a subset of $\mathcal{C}(F)$, $F \subseteq \mathcal{C}(F)$: we have "enlarged" F.)

3. Prove that the sum and product of Cauchy sequences is Cauchy.

4. Let $[a_n]$ denote the equivalence class containing the Cauchy sequence $\{a_n\}$. Given Cauchy sequences $\{a_n\}$ and $\{b_n\}$, define the sum and product of the equivalence classes containing them by

$$[a_n] +_{\mathcal{C}(F)} [b_n] := [a_n + b_n]$$

$$[a_n] \cdot_{\mathcal{C}(F)} [b_n] := [a_n b_n]$$

Prove that these rules are well-defined by showing that if $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$, then $\{a_n + b_n\} \sim \{a'_n + b'_n\}$ and $\{a_n b_n\} \sim \{a'_n b'_n\}$

5. If $\mathcal{C}(F)$ denotes the set of equivalence classes of Cauchy sequences in F, then with the sum and product operations in 3. $\mathcal{C}$ is in fact a field in such a way that the "copy" of F in $\mathcal{C}(F)$ in 2. above is the field F we started with: $F \subseteq \mathcal{C}(F)$ is a sub*field*. Don't try to prove this, but identify the additive and multiplicative identities 0 and 1 in $\mathcal{C}(F)$ and verify that $[a_n] +_{\mathcal{C}(F)} 0 = [a_n]$ and $[a_n] \cdot_{\mathcal{C}(F)} 1 = [a_n]$ for all Cauchy sequences $\{a_n\}$.(Keep in mind that your choice of 0 (or 1) in your answer will be an *equivalence class* of Cauchy sequences. This class may be identified by specifying *any* Cauchy sequence in it.)