Math 431 January 19, 2018

Assignment 2
(Due February 2)

Reading: (*from Reed*) §2.1, 2.2

Problems: §1.1: #3, 4, 7, 8, 10, 11 (in #3 and #4 replace "real numbers" with "elements of a field"; in the rest assume only that x, y, a and b are elements of an ordered field F, except that in #11 assume in addition that F is Archimedean. In the hint for #11, use WO to show m exists)
§1.4: #9, 11 (You don't need *a.* to do *c.*; try using *b.* and #11 above)

Additional Problems: 1. Let F be a field. Using only the axioms (those on p.1 of the text), prove that $-ab = (-a)b$ for all a, $b \in F$.

2. Prove that the field $\mathbb{C}$ of complex numbers cannot be given the structure of an ordered field. (*Suggestion:* Argue by contradiction: suppose a subset $P \subseteq \mathbb{C}$ exists with the required properties; then $i \in P \cup (-P)$, where i is the complex number such that $i^2 = -1$. Deduce the contradiction from this.)

3. Let F be a field. Prove that if there is an integer $n \in \mathbb{N}$ such that $1 + 1 + \cdots + 1$ (n terms) $= 0$, then there is no subset $P \subseteq F$ saisfying the axioms of an odered field. (It can be deduced from this that if (F, P) is an ordered field, then $\mathbb{Q} \subseteq F$.) Use this to prove that no finite field can be given the structure of an ordered field.

4. With the definition of P given in class, prove that $\mathbb{Q}(t)$ is an ordered field. To show that if $\frac{p(t)}{q(t)}, \frac{r(t)}{s(t)} \in P$, then $\frac{p(t)}{q(t)} + \frac{r(t)}{s(t)} \in P$, you may do only this case: assume $p(t) = a_n t^n + \cdots$, $q(t) = b_m t^m + \cdots$, $r(t) = c_k t^k + \cdots$ and $s(t) = d_\ell t^\ell + \cdots$, where $n + \ell > k + m$.

5. Prove that the Archimedean property does not hold in the ordered field $\mathbb{Q}(t)$, by considering its two elements $\frac{1}{1}$ and $\frac{t^2}{1}$.

6. Let $f : \mathbb{N} \to \mathbb{Z}$ be the function

$$f(n) := \begin{cases} \frac{n}{2}, n \text{ even} \\ \frac{1-n}{2}, n \text{ odd} \end{cases}$$

Prove that f is is bijective by showing two things: if $m \in \mathbb{Z}$, there $n \in \mathbb{N}$ such that $f(n) = m$; and if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

7. Prove that the composition of injections is an injection and the composition of surjections is a surjection. (Therefore the composition of bijections is a bijection.)

8. All horses are the same color: clearly, any set of one horse is the same color; assuming that in every set of n horses all are the same color, we conclude that every set of $n + 1$ horses, labeled from 1 to $n + 1$, has the same color, by considering the subsets of horses labeled from 1 to n and from 2 to $n + 1$, each of which must be the same color. Where's

the flaw in this argument? (One possibility is that Mathematical Induction, hence WO, is flawed, or can't be applied here for some reason.)

We said in class that we would assume whatever properties we need for the rational numbers $\mathbb{Q}$. However, these are are all easily derivable from properties of the integers $\mathbb{Z}$: $\mathbb{Q}$ is *the field of fractions* of $\mathbb{Z}$. Here is one example of this.

9. Prove that $\mathbb{Q}$ is Archimedean. (*Suggestion:* Given x, $y \in \mathbb{Q}$, both > 0, we want to show that there is $n \in \mathbb{N}$ such that $nx > y$, or $n > y/x$. Write $y/x = r/s$ where r, $s \in \mathbb{N}$ and give a specific n that works.)