Math 139 September 1, 2009

Assignment 2
(Due September 8)

Reading: (*from Reed*) §2.1, 2.2

Problems: §1.1: #8, 10, 11 (in the hint, use WO to show m exists)
§1.4: #9, 11 (You don't need *a.* to do *c.*; try using *b.* and #11 above)

Additional Problems: 1. Prove that the field $\mathbb{C}$ of complex numbers cannot be given the structure of an ordered field. (*Suggestion:* Argue by contradiction: suppose a subset $P \subseteq \mathbb{C}$ exists with the required properties; then $i \in P \cup (-P)$, where i is the complex number such that $i^2 = -1$. Deduce the contradiction from this.)

2. Let F be a field. Prove that if there is an integer $n \in \mathbb{N}$ such that $1 + 1 + \cdots + 1$ (n terms) $= 0$, then there is no subset $P \subseteq F$ saisfying the axioms of an odered field. (It can be deduced from this that if (F, P) is an ordered field, then $\mathbb{Q} \subseteq F$.) Use this to prove that no finite field can be given the structure of an ordered field.

3. Prove that the Archimedean property does not hold in the ordered field $\mathbb{R}(x)$, by considering its two elements $\frac{1}{1}$ and $\frac{x^2}{1}$.

4. Since $a_n = n(n+1)/2$, $n = 1, 2, \ldots$, is an increasing sequence of positive integers, every integer is in one and only one of the subsets I_n of $\mathbb{N}$ where

$$I_1 = \{1\}, \quad I_n = (a_{n-1}, a_n] \cap \mathbb{N} = \{n(n-1)/2 + 1, \ldots, n(n+1)/2\}, n \geq 2$$

Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be the function defined by the following rule. Let $\ell \in I_n$, say $\ell = n(n-1)/2 + k$, where $k \in \{1, \ldots, n\}$. Then define $f(\ell) = (n - k + 1, k)$. Prove that f is bijective.

5. All horses are the same color: clearly, any set of one horse is the same color; assuming that in every set of n horses all are the same color, we conclude that every set of $n + 1$ horses, labeled from 1 to $n + 1$, has the same color, by considering the subsets of horses labeled from 1 to n and from 2 to $n + 1$, each of which must be the same color. Where's the flaw in this argument? (One possibility is that Mathematical Induction, hence WO, is flawed.)