1. Uniqueness for the linear equation.

Suppose

$$a : \Omega \times (0,T) \to \mathbf{Sym}(\mathbf{R}^n);$$
$$b : \Omega \times (0,T) \to \mathbf{R}^n;$$
$$c : \Omega \times (0,T) \to \mathbb{R}$$

and

$$\alpha, \beta, \gamma$$

are nonnegative real numbers such that, for any $(x,t) \in \Omega \times (0,T)$,

$$a(x,t)(v) \bullet v \geq \alpha |v|^2 \quad \text{whenever } v \in \mathbf{R}^n;$$
$$|b(x,t)| \leq \beta;$$
$$|c(x,t)| \leq \gamma.$$

Suppose $u : \Omega \times (0,T) \to \mathbb{R}$ is such that

(1) $$u(b,t) = 0 \quad \text{whenever } (b,t) \in \partial\Omega \times (0,T);$$

(2) $$\liminf_{t \downarrow 0} \int_\Omega u^2 = 0;$$

and

(3) $$\frac{\partial u}{\partial t} = \mathbf{div}\,(a(\nabla u)) + b \bullet u + c\,u.$$

We will show that

(4) $$u(x,t) = 0 \quad \text{for } (x,t) \in \Omega \times (0,T).$$

Suppose $0 < \eta < \infty$ and

$$\eta^2 < \alpha.$$

Then

(5) $$\beta\,|u|\,|\nabla u| = 2(\eta|\nabla u|)\left(\frac{\beta\,|u|}{2\eta}\right) \leq \eta^2 |\nabla u|^2 + \frac{\beta^2}{4\eta^2} u^2.$$

In view of (3) and (5) we find that

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{u^2}{2} &= u\frac{\partial u}{\partial t} \\
&= u\,\mathbf{div}\,(a(\nabla u)) + u\,b \bullet \nabla u + cu^2 \\
&= \mathbf{div}\,(u\,a(\nabla u)) - \nabla u \bullet a(\nabla u) + u\,b \bullet \nabla u + cu^2 \\
&\leq \mathbf{div}\,(u\,a(\nabla u)) - \alpha|\nabla u|^2 + \beta\,|u|\,|\nabla u| + \gamma u^2 \\
&\leq \mathbf{div}\,(u\,a(\nabla u)) + (\eta^2 - \alpha|\nabla u|^2) + \left(\frac{\beta^2}{4\eta^2} + \gamma\right) u^2 \\
&\leq \mathbf{div}\,(u\,a(\nabla u)) + (2\eta\beta - \alpha|\nabla u|^2) + \left(\frac{\beta^2}{4\eta^2} + \gamma\right) u^2 \\
&\leq \mathbf{div}\,(u\,a(\nabla u)) + \left(\frac{\beta^2}{4\eta^2} + \gamma\right) u^2.
\end{aligned}$$

Integrating over Ω and invoking (1) we find that

$$\frac{d}{dt}\int_\Omega \frac{u^2}{2} \leq 2\left(\frac{\beta^2}{4\eta^2} + \gamma\right)\int_\Omega \frac{u^2}{2}.$$

That (4) holds now follows from (2) and the following Lemma.

Lemma 1.1. Suppose $f : (0, T) \to [0, \infty)$ is continuously differential, $0 \le M < \infty$,

$$\dot{f}(t) \le M f(t) \quad \text{whenever } 0 < t < T$$

and

$$\liminf_{t \downarrow 0} f(t) = 0.$$

Then

$$f(t) = 0 \quad \text{for } 0 < t < T.$$

Proof. Suppose, contrary to the Lemma, $0 < t_0 < T$ and $f(t_0) > 0$. Let I be the connected component of t_0 in $\{t \in I : f(t) > 0\}$.

Since

$$\frac{d}{dt} \ln f(t) = \frac{\dot{f}(t)}{f(t)} \le M \quad \text{whenever } t \in I$$

we find that

$$\ln \left(\frac{f(t)}{f(t_0} \right) \le M(t - t_0)$$

or

$$f(t) \le f(t_0) e^{M(t-t_0)} \quad \text{for } t \in I.$$

Letting $t_0 \downarrow \inf I$ we infer that $f(t) = 0$ for $t \in I$. □