1. Uniform convergence.

Suppose X is a set and (Y,σ) is a metric space. We let

$$\mathcal{B}(X,Y)$$

be the set of bounded functions from X to Y; that is, $f \in \mathcal{B}(X,Y)$ if $f : X \to Y$ and $\mathbf{diam}\,\mathbf{rng}\, f < \infty$. For each $f,g \in \mathcal{B}(X,Y)$ we set

$$\Sigma(f,g) = \sup\{\sigma(f(x),g(x)) : x \in X\}.$$

Proposition 1.1. Σ is metric on $\mathcal{B}(X,Y)$.

Proof. Suppose $f,g \in \mathcal{B}(X,Y)$ and $a \in X$. Then

$$\begin{aligned}\sigma(f(x),g(x)) &\le \sigma(f(x),f(a)) + \sigma(f(a),g(a)) + \sigma(g(a),g(x))\\ &\le \mathbf{diam}\,\mathbf{rng}\, f + \sigma(f(a),g(a)) + \mathbf{diam}\,\mathbf{rng}\, g\end{aligned}$$

for any $x \in X$. Thus $\Sigma(f,g) < \infty$. It is evident that $\Sigma(g,f) = \Sigma(f,g)$ and that if $\Sigma(f,g) = 0$ then $f = g$.

Suppose $f,g,h \in \mathcal{B}(X,Y)$. Then

$$\sigma(f(x),h(x)) \le \sigma(f(x),g(x)) + \sigma(g(x),h(x)) \le \Sigma(f,g) + \Sigma(g,h)$$

for any $x \in X$ from which we conclude that $\Sigma(f,g) \le \Sigma(f,g) + \Sigma(g,h)$. □

Example 1.1. Suppose Y is a vector space normed by $|\cdot|$ and σ is the corresponding metric. Note that

$$\mathcal{B}(X,Y)$$

is then the set of functions $f : X \to Y$ such that

$$\sup\{|f(x)| : x \in A\} < \infty.$$

We set

$$||f|| = \sup\{|f(x)| : x \in X\} \text{ whenever } f \in \mathcal{B}(X,Y)$$

and note that

$$\Sigma(f,g) = ||f-g|| \text{ whenever } f,g \in \mathcal{B}(A,Y).$$

Obviously,

$$||f|| = 0 \;\Leftrightarrow\; f = 0 \quad \text{whenever } f \in \mathcal{B}(X,Y).$$

If $c \in \mathbb{R}$ and $f \in \mathcal{B}(X,Y)$ we have

$$||cf|| = \{|(cf)(x)| : x \in X\} = \{|c||f(x)| : x \in X\} = |c|\{|f(x)| : c \in X\} = |c|||f||.$$

Moreover,

$$||f+g|| = \sup\{|f(x)+g(x)| : x \in X\} \le ||f|| + ||g||$$

whenever $f,g \in \mathcal{B}(X,Y)$. In particular, $\mathcal{B}(X,Y)$ is a linear subspace of Y^X. Thus $\mathcal{B}(X,Y)$ is a normed vector space with respect to $||\cdot||$.

Proposition 1.2. Σ is complete if σ is complete.

Proof. Suppose f is a Cauchy sequence in $\mathcal{B}(X,Y)$ with respect to Σ. Then, for each $x \in X$, $\mathbb{N} \ni \nu \mapsto f_\nu(x)$ is a Cauchy sequence in Y and so, as σ is complete, converges to some $g(x)$.

We now show that $\lim_{\nu\to\infty} \Sigma(f_\nu,g) = 0$. Suppose $0 < \eta < \epsilon < \infty 0$. Let $N \in \mathbb{N}$ be such that

$$\mu,\nu \in \mathbb{N} \text{ and } \mu,\nu \ge N \;\Rightarrow\; \Sigma(f_\mu,f_\nu) < \eta.$$

Then for any $x \in X$ and any $\mu, \nu \in \mathbb{N}$ with $\mu, \nu \geq N$ we have

$$\begin{aligned}\sigma(f_\mu(x), g(x)) &\leq \sigma(f_\mu(x), f_\nu(x)) + \sigma(f_\nu(x), g(x)) \\ &\leq \Sigma(f_\mu, f_\nu) + \sigma(f_\nu(x), g(x)) \\ &< \eta + \sigma(f_\nu(x), g(x)).\end{aligned}$$

Letting $\nu \to \infty$ we find that if $\mu \geq N$ then

$$\sigma(f_\mu(x), g(x)) \leq \eta \quad \text{for any } x \in X$$

which implies $\Sigma(f_\mu, g) \leq \eta < \epsilon$. That is, $f_\nu \to g$ as $\nu \to \infty$ with respect to Σ, as desired. □

Now suppose X is a topological space. Let

$$\mathcal{C}(X, Y) = \{f \in \mathcal{B}(X, Y) : f \text{ is continuous}\}.$$

Theorem 1.1. $\mathcal{C}(X, Y)$ is a closed subset of $\mathcal{B}(X, Y)$.

Remark 1.1. It follows that $\mathcal{C}(X, Y)$ is complete with respect to the metric on it induced by Σ provided Y is complete.

Proof. Suppose $g \in \mathbf{cl}\, \mathcal{C}(X, Y)$.

Suppose $a \in X$ and let $\epsilon > 0$. Since $g \in \mathbf{cl}\, \mathcal{C}(X, Y)$ we there is $f \in \mathbf{U}(g, \epsilon/3) \cap F$. Since f is continuous at a there is an open subset U of X such that

$$x \in U \;\Rightarrow\; \sigma(f(x), f(a)) \leq \epsilon/3.$$

Then

$$\begin{aligned}\sigma(g(x), g(a)) &\leq \sigma(g(x), f(x)) + \sigma(f(x), f(a)) + \sigma(f(a), g(a)) \\ &\leq \Sigma(f, g) + \epsilon/3 + \Sigma(f, g) \\ &\leq \epsilon.\end{aligned}$$

So g is continuous and, therefore, $\mathcal{C}(X, Y)$ is a closed subset of $\mathcal{B}(X, Y)$. □

Remark 1.2. Suppose X is compact and let

$$\mathcal{K}(X, Y) = \{f \in Y^X : f \text{ is continuous}\}.$$

Then

$$\mathcal{K}(X, Y) \subset \mathcal{B}(X, Y).$$

If Y is complete then $\mathcal{K}(X, Y)$ is also complete by virtue of the preceding Theorem. In particular, if Y is a Banach space so is $\mathcal{K}(X, Y)$.

Remark 1.3. For each $\nu \in \mathbf{N}$ let $f_\nu(x) = x^\nu$, $0 \leq x \leq 1$. Evidently,

$$\lim_{\nu \to \infty} f_\nu(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Thus the pointwise limit is not continuous and, therefore, the convergence is not uniform. Indeed, if $\mu, \nu \in \mathcal{N}$ and $\nu > \mu$ then

$$(f_\mu - f_\nu)(x) = x^\mu(1 - x^{\nu - \mu}) \to 1 \quad \text{as } n \to \infty$$

which implies that

$$\lim_{\nu \to \infty} ||f_\mu - f_\nu|| = 1 \quad \text{for any } \mu \in \mathbb{N}.$$