Inscribing triangles to compute area.

Proposition. Suppose $k, l \in \mathbf{L}(\mathbf{R}^m, \mathbf{R}^n)$ and $v_1, \dots, v_m \in \mathbf{R}^m$. Then

$$\left(|\bigwedge\nolimits_m k + l\right)(v_1 \wedge \cdots \wedge v_m) - \left(\bigwedge\nolimits_m l\right)(v_1 \wedge \cdots \wedge v_m)| \le \sum_{p=1}^{m} \binom{m}{p} ||l||^{m-p} ||k||^p \, |v_1| \cdots |v_m|.$$

Proof. For each $\lambda \in \Lambda(p, m)$ let $\hat{\lambda} \in \Lambda(m-p, m)$ be such that $\mathbf{rng}\,\hat{\lambda} = \{1, \dots, m\} \sim \mathbf{rng}\,\lambda$ and let $\mathbf{s}(\lambda) \in \{\pm 1\}$ be such that

$$v_1 \wedge \cdots \wedge v_m = \mathbf{s}(\lambda) v_\lambda \wedge v_{\hat{\lambda}}.$$

We have

$$\begin{aligned}\left(\bigwedge\nolimits_m k + l\right)(v_1 \wedge \cdots \wedge v_m) &= \sum_{p=0}^{m} \sum_{\lambda \in \Lambda(p,m)} \mathbf{s}(\lambda) \left(\bigwedge\nolimits_{m-p} l\right)(v_\lambda) \wedge \left(\bigwedge\nolimits_p k\right)(v_{\hat{\lambda}}) \\ &= \left(\bigwedge\nolimits_m l\right)(v_1 \wedge \cdots \wedge v_m) \\ &\quad + \sum_{p=1}^{m} \sum_{\lambda \in \Lambda(p,m)} \mathbf{s}(\lambda) \left(\bigwedge\nolimits_{m-p} l\right)(v_\lambda) \wedge \left(\bigwedge\nolimits_p k\right)(v_{\hat{\lambda}}).\end{aligned}$$

□

Definition. Whenever p is a positive integer and $a_0, a_1, \dots, a_p \in \mathbf{R}^n$ we let

$$[a_0, a_1, \dots, a_p] = \{\sum_{i=0}^{p} c_i a_i : 0 \le c_i \le 1, \ i = 0, 1, \dots, p, \text{ and } \sum_{i=0}^{p} c_i = 1\}$$

and call this set the **p-simplex spanned by** $a_0, a_1, \dots, a_p$.

Proposition. Suppose U is a convex open subset of $\mathbf{R}^m$,

$$f : U \to \mathbf{R}^m,$$

f is continuously differentiable,

$$a_0, a_1, \dots a_m \in U,$$

$$S = [a_0, a_1, \dots, a_m]$$

and

$$S_f = [f(a_0), f(a_1), \dots, f(a_m)].$$

Then

$$\Big| J_m f(a) ||S||_m - ||S_f||_m \Big| \le \Big(\sum_{p=1}^{m} \binom{m}{p} ||\partial f(a)||^{m-p} \epsilon^p\Big) |a_1 - a_0| \cdots |a_m - a_0|$$

where

$$\epsilon = \sup\{||\partial f(x) - \partial f(a)|| : x \in S\}.$$

Proof. Set

$$r(a, x) = \int_0^1 \partial f((1-t)a + tx) - \partial f(a)\, dt, \quad a, x \in U.$$

Note that

$$f(x) = f(a) + \partial f(a)(x - a) + r(a, x)(x - a), \quad a, x \in U.$$

Now apply the previous Proposition. □

Theorem. Suppose U is a convex open subset of $\mathbf{R}^m$,

$$f : U \to \mathbf{R}^n$$

f is continuously differentiable and K is a compact subset of U which is the union of a finite family of nonoverlapping m-simplices.

Then for any $\theta > 0$ and $\epsilon > 0$ there is $\delta > 0$ such that

$$\Big| \int_K J_m f(x)\, dx - \sum_{S \in \mathcal{S}} ||S_f||_m \Big| < \epsilon$$

whenever $\mathcal{S}$ is a family of nonoverlapping m-simplices with union K satisfying

$$\mathbf{diam}\, S < \epsilon \quad \text{and} \quad \frac{||S||_m}{\mathbf{diam}\, S^m} > \theta$$

and where, for each $S \in \mathcal{S}$, $S_f = [f(a_0), f(a_1), \ldots, f(a_m)]$ if $S = [a_0, a_1, \ldots, a_m]$
Proof. Combine the above with the fact that ∂f is uniformly continuous on K. $\square$

An example illustrating why the hypotheses in the previous Theorem are necessary. Let

$$f : \mathbf{R}^2 \to \mathbf{R}^3$$

be such that

$$f(\theta, z) = U(\theta) + z\mathbf{e}_3, \quad (\theta, z) \in \mathbf{R}^2,$$

where for $\theta \in \mathbf{R}$ we have set

$$U(\theta) = (0, \cos\theta, \sin\theta) \in \mathbf{R}^3.$$

For $0 < h < \pi$ and $k > 0$ let $T_{h,k}$ be the triangle with vertices

$$f(-h, 0),\ f(0, k),\ f(h, 0).$$

The square of twice the area $T_{h,k}$ is

$$\begin{aligned} &|(f(-h,0) - f(0,k)) \wedge (f(h,0) - f(0,k))|^2 \\ &\qquad = |U(h) \wedge U(-h) + (U(h) - U(-h)) \wedge k\mathbf{e}_3|^2 \\ &\qquad = |U(h) \wedge U(-h)|^2 + k^2 |U(h) - U(-h)|^2. \end{aligned}$$

Thus the twice the area of $T_{h,k}$ tends to $|U(h) \wedge U(-h)|$ as $k \downarrow 0$ and so the ratio of the area of $T_{k,h}$ to the area of the triangle with vertices $(-h, 0), (0, k), (h, 0)$ tends to infinity as $k \downarrow 0$.

For a triangle T in $\mathbf{R}^2$ we let T_f be the triangle in $\mathbf{R}^3$ whose vertices are the image under f of the vertices of T. If we were to define the area of $f[(-\pi, \pi) \times (0, 1)]$ in a fashion similar to the way the length of a curve is typically defined we get the wrong answer because

$$\sup\{\sum_{T \in \mathcal{T}} |T_f| : \mathcal{T} \text{ is a finite nonoverlapping family of triangles in } [-\pi, \pi] \times [0, 1]\} = \infty.$$

This situation is not remedied by requiring the diameters of the inscribed triangles to be small. The problem occurs when the ratio of the square of the diameter of a triangle is large compared to the area of the triangle.