1. Fubini's Theorem.

Suppose $n, m \in \mathbb{N}^+$ and $0 < m < n$. We identify $\mathbb{R}^n$ with $\mathbb{R}^m \times \mathbb{R}^{n-m}$ in the natural way. For each function g with domain $\mathbb{R}^n$ and each $y \in \mathbb{R}^{n-m}$ we define the function $\mathbf{s}_y(g)$ with domain $\mathbb{R}^m$ by letting $\mathbf{s}_y(g)(x) = g(x, y)$ for $x \in \mathbb{R}^m$.

Lemma 1.1. Suppose $f \in \mathcal{F}_n^+$ and

$$F(y) = \mathbf{l}^+_{n-m}(\mathbf{s}_y(f)) \quad \text{for } y \in \mathbb{R}^{n-m}.$$

Then

$$\mathbf{l}^+_{n-m}(F) \le \mathbf{l}^+_n(f).$$

Proof. Suppose $s \in S^+_{n,\uparrow}$ and $f \le \sup s$. Let S be the sequence such that, for each $\nu \in \mathbb{N}$, $S_\nu : \mathbb{R}^{n-m} \to [0, \infty)$ and and $S_\nu(y) = I_m(\mathbf{s}_y(s\nu))$. Then $S \in S^+_{n-m,\uparrow}$ and $F \le \sup S$. It follows that

$$\mathbf{l}^+_{n-m}(F) \le I^+_{S,\uparrow} = I^+_n(s);$$

the Lemma follows. □

For each $y \in \mathbb{R}^{n-m}$ we define

$$S_y : \mathcal{F}_n \to \mathcal{F}_m$$

by letting

$$S_y(f)(x) = f(\alpha(x) + \beta(y)) \quad \text{whenever } f \in \mathcal{F}_n \text{ and } x \in \mathbb{R}^m.$$

One may easily check that if $s \in \mathcal{S}_n$ then

(1) $S_y(s) \in \mathcal{S}_m$ whenever $y \in \mathbb{R}^{n-m}$;
(2) $\mathbb{R}^{n-m} \ni y \mapsto I_m(S_y(s)) \in \mathcal{S}_{n-m}$;
(3) $I_m(\mathbb{R}^{n-m} \ni y \mapsto I_m(S_y(s))) = I_n(s)$.

Theorem 1.1.

$$|\{y \in \mathbb{R}^{n-m} : S_y(f) \notin \mathcal{L}_m\}|^* = 0$$

and if

$$F : \mathbb{R}^{n-m} \to \mathbb{R}$$

is such that

$$F(y) = \mathbf{L}_m(\mathbf{s}_y(f)) \quad \text{whenever } \mathbf{s}_y(f) \in \mathcal{L}_m$$

then $F \in \mathcal{L}_{n-m}$ and

$$\mathbf{L}_n(f) = \mathbf{L}_{n-m}(F).$$

Proof. **Part One** Choose sequences ϵ and η in $(0, \infty)$ such that $\lim_{\nu \to \infty} \epsilon_\nu = 0$ and $\sum_{\nu=1}^\infty \eta_\nu < \infty$.

Next, choose a sequence s in $\mathcal{S}_n$ such that $\mathbf{l}^+(|f - s_\nu|) \le \epsilon_\nu \eta_\nu$.

For each $y \in \mathbb{R}^{n-m}$ and each $g : \mathbb{R}^n \to \mathbb{R}$ let

$$\mathbf{s}_y(g) : \mathbb{R}^m \to \mathbb{R}$$

be such that $\mathbf{s}_y(g)(x) = g(x, y)$ for $x \in \mathbb{R}^n$.

For each $\nu \in \mathbb{N}$ let

$$E_\nu = \{y \in \mathbb{R}^{n-m} : \mathbf{l}^+_m(|\mathbf{s}_y(f) - \mathbf{s}_y(s_\nu))|) \le \epsilon_\nu\}.$$

let

$$D = \cup_{N=0}^\infty \cap_{\nu=N}^\infty E_\nu.$$

Part Two $y \in D \;\Rightarrow\; \mathbf{s}_y(f) \in \mathcal{L}_m$.

Suppose $y \in D$. Choose a positive integer N such that $y \in \bigcap_{\nu=N}^{\infty} E_\nu$. Then for any $\nu \in \mathbb{N}$ with $\nu \geq N$ we have $\mathbf{l}_m^+(|\mathbf{s}_y(f) - \mathbf{s}_y(s_\nu)|)| \leq \epsilon_\nu$ which implies $\mathbf{s}_y(f) \in \mathbf{Leb}_{n-m}$.

Part Three $|\mathbb{R}^n \sim D|^* = 0$.

For each $\nu \in \mathbb{N}$ we have

$$|\mathbb{R}^n \sim E_\nu|^* \leq \frac{1}{\epsilon_\nu}\mathbf{l}^+(S_x(m_\nu)) \leq \eta_\nu;$$

Conseqently,

$$|\mathbb{R}^n \sim D|^* \leq \inf_N \left| \bigcup_{i=N}^{\infty} \mathbb{R}^n \sim E_i \right|^* \leq \inf_N \sum_{i=N}^{\infty} |\mathbb{R}^n \sim E_i|^* \leq \inf_N \sum_{i=N}^{\infty} \eta_i = 0.$$

Now $|F - G| \leq \infty 1_{\mathbb{R}^n - E}$; combining this with the Lemma we find that

$$\mathbf{l}_{n-m}^+(|G - S_\nu|) = \mathbf{l}_{n-m}^+(|F - S_\nu|) \leq \mathbf{l}_n^+(f - s_\nu);$$

since $\mathbf{l}_{n-m}^+(S_\nu) = \mathbf{l}_n^+(s_\nu)$ for each $\nu \in \mathbb{N}$ the Theorem follows.

$\square$