Tangency.

Let X be a normed vector space.

Definition. Suppose $v \in X$ and $C \subset X$. We say C **is a cone with vertex** v if

$$x \in C \sim \{v\} \text{ and } t \geq 0 \;\Rightarrow\; v + t(x - v) \in C.$$

Note that the empty set is a cone with vertex v and that $v \in C$ if $C \sim \{v\} \neq \emptyset$.
Proposition. Suppose $v \in X$ and $\mathcal{C}$ is a nonempty family of cones with vertex v. Then $\cup\mathcal{C}$ is a cone with vertex v.
Proof. This is immediate. □
Proposition. Suppose $v \in X$ and C is a cone with vertex v. Then the closure of C is a cone with vertex v.
Proof. Exercise. □
Definition. Suppose $A \subset X$, $a \in \mathbf{acc}\, A$. For each $\delta > 0$ we let

$$\mathbf{Tan}_a(A, \delta) = \mathbf{cl}\,\{t(x - a) : t \geq 0, \text{ and } x \in (A \sim \{a\}) \cap \mathbf{B}_a(\delta)\}.$$

Note that. by virtue of the previous Proposition, $\mathbf{Tan}_a(A, \delta)$ is a closed cone with vertex 0.

We let

$$\mathbf{Tan}_a(A) = \bigcap_{\delta>0} \mathbf{Tan}_a(A, \delta)$$

and we let

$$\mathbf{Nor}_a(A) \;=\; \{\omega \in X^* : \omega(v) \leq 0 \text{ whenever } v \in \mathbf{Tan}_a(A)\}.$$

Note that $\mathbf{Tan}_a(A)$ and $\mathbf{Nor}_a(A)$ are closed cones in X and X^*, respectively, by virtue of the first Proposition above..

In case X is an inner product space we will also let

$$\mathbf{Nor}_a(A) \;=\; \{w \in X : v \bullet w \leq 0 \text{ whenever } v \in \mathbf{Tan}_a(A)\}$$

and rely on the context to resolve the ambiguity.

Theorem. Suppose X is finite dimensional, $A \subset X$, $a \in \mathbf{acc}\, A$. Then $\mathbf{Tan}_a(A) \neq \emptyset$. Moreover, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mathbf{cl}\, A \cap \mathbf{B}_a(\delta) \subset a + \{v \in X : \mathbf{dist}(v, \mathbf{Tan}_a(A)) \leq \epsilon|v|\}$$

Proof. Let $K = \{u \in X : |u| = 1\}$ and note that K is compact because X is finite dimensional. Let $L = K \cap \mathbf{Tan}_a(A)$ and, for each $\delta > 0$, let $T_\delta = K \cap \mathbf{Tan}_a(A, \delta)$. Then $\{T_\delta : \delta > 0\}$ is a nonempty nested family of closed subsets of the compact set K whose nonempty intersection is L. Moreover, if U is an open set containing L then there is $\delta > 0$ such that $T_\delta \subset U$.

Now suppose $\epsilon > 0$. Let

$$U = \{v \in X \sim \{0\} : \mathbf{dist}(v, \mathbf{Tan}_a(A)) < \epsilon|v|\}$$

and note that U is open. Since $L \subset U$ and U is open there is $\delta > 0$ such that $T_\delta \subset U$. □

Proposition. Suppose $A \subset X$, $a \in \mathbf{acc}\, A$ and $v \in X \sim \{0\}$. The following are equivalent.

(i) $v \in \mathbf{Tan}_a(A)$.

(ii) For each $\epsilon > 0$ and $\delta > 0$ there are $s > 0$ and $x \in (A \sim \{a\}) \cap \mathbf{B}_a(\delta)$ such that

$$|(x-a) - sv| \le \epsilon |x-a|.$$

Proof. Suppose $v \in \mathbf{Tan}_a(A)$, $\epsilon > 0$ and $\delta > 0$. Let η be such that $0 < \eta < 1$ and $\frac{1}{1-\eta} \le \epsilon$. Since v is a member of the closure of $\mathbf{Tan}_a(A,\delta)$ there are $x \in (A \sim \{a\}) \cap \mathbf{B}_a(\delta)$ and $t \ge 0$ such that $|t(x-a) - v| \le \eta |v|$. This implies $\big| t|x-a| - |v| \big| \le \eta|v|$ so that $t|x-a| \ge (1-\eta)|v|$. In particular, $t|x-a| > 0$. Let $s = \frac{1}{t}$. Then

$$\big|(x-a) - sv\big| = \frac{1}{t}|t(x-a) - v| \le \frac{|x-a|}{(1-\eta)|v|}\eta|v| \le \epsilon|x-a|$$

so (ii) holds.

On the other hand, suppose (ii) holds, let $\delta > 0$ and let $\rho > 0$. Let ζ be such that $0 < \zeta < 1$ and $\frac{\zeta|v|}{1-\zeta} \le \rho$. Let $s > 0$ and $x \in (A \sim \{a\}) \cap \mathbf{B}_a(\delta)$ such that $|(x-a) - sv| \le \zeta|x-a|$. Then $\big||x-a| - s|v|\big| \le \zeta|x-a|$ so $s|v| \ge (1-\zeta)|x-a|$. Set $t = \frac{1}{s}$. Then

$$|t(x-a) - v| = \frac{1}{s}|(x-a) - sv| \le \frac{|v|}{(1-\zeta)|x-a|}\zeta|x-a| \le \rho.$$

Owing to the arbitrariness of ρ we infer that $v \in \mathbf{Tan}_a(A,\delta)$. Owing to the arbitrariness of δ we infer that (i) holds.

Theorem. Suppose X and Y are normed spaces, $A \subset X$, $a \in \mathbf{int}\, A$, $f : A \to Y$ and f is differentiable at a. Then

$$\mathbf{rng}\, \partial f(a) \sim \{0\} \subset \mathbf{Tan}_{f(a)}(f[A]).$$

Proof. Suppose $v \in X$ and $w = \partial f(a)(v) \ne 0$. Let $\epsilon > 0$ and choose η such that $0 < |v|\eta < |w|$ and $\frac{\eta}{|w| - \eta|v|} \le \epsilon|v|$. Choose $\delta > 0$ such that

$$x \in A \cap \mathbf{B}_a(\delta) \;\Rightarrow\; |f(x) - f(a) - \partial f(a)(x-a)| \le \eta|x-a|.$$

If $t > 0$ and $t|v| \le \delta$ we have $|f(a+tv) - f(a) - tw| \le \eta t|v|$ so $|f(a+tv) - f(a)| \ge t(|w| - \eta|v|)$. Consequently,

$$|f(a+tv) - f(a) - tw| \le \frac{t\eta|v|}{|f(a+tv) - f(a)|}|f(a+tv) - f(a)| \le \frac{\eta|v|}{|w| - \eta|v|}|f(a+tv) - f(a)| \le \epsilon|f(a+tv) - f(a)|.$$

The Theorem now follows from a previous Proposition. $\square$

Theorem. Suppose X and Y are normed spaces, X is finite dimensional, A is an open subset of X, f is differentiable at each point of A and $b \in \mathbf{rng}\, f$.

Suppose, additionally, that

(i) $\mathbf{ker}\, \partial f(a) = \{0\}$ whenever $a \in A$ and $f(a) = b$;

(ii) there is $s > 0$ such that $f^{-1}[\mathbf{B}_b(s)]$ is a compact subset of A.

Then $b \in \mathbf{acc}\, \mathbf{rng}\, f$, $\{a \in A : f(a) = b\}$ is finite and

$$\mathbf{Tan}_b(\mathbf{rng}\, f) = \bigcup \{\mathbf{rng}\, \partial f(a) : a \in A \text{ and } f(a) = b\}. \tag{1}$$

Proof. We have already shown that the right hand side of (1) is a subset of the left hand side. So suppose $w \in \mathbf{Tan}_b(\mathbf{rng}\, f)$, $|w| = 1$ and $\epsilon > 0$. We will obtain $a \in A$ and $v \in X$ such that $f(a) = b$ and $|w - \partial f(a)(v)| \le \epsilon$. This will show that w is a point of the closure of the range of $\partial f(a)$. Since X is finite dimensional, the range of $\partial f(a)$ is closed so the proof will be complete.

Let $K = \{a \in A : f(a) = b\}$. K is closed relative to A because f is continuous. Since K is a subset of the compact set $f^{-1}[\mathbf{B}_b(s)]$ we infer that K is compact. For each $a \in K$ choose m_a, M_a such that $0 < m_a \le M_a < \infty$ and

$$m_a|v| < |\partial f(a)(v)| < M_a|v| \quad \text{whenever } v \in X \sim \{0\};$$

this is possible because X is finite dimensional and $\mathbf{ker}\,\partial f(a) = \{0\}$. For any $a, x \in A$ we have

$$||f(x) - f(a)| - |\partial f(a)(x-a)|| \le |f(x) - f(a) - \partial f(a)(x-a)|;$$

it follows that for each $a \in K$ there is $\rho_a > 0$ such that $\mathbf{B}_a(\rho_a) \subset X$ and

$$m_a|x-a| \le |f(x) - f(a)| \le M_a|x-a| \quad \text{whenever } x \in \mathbf{B}_{\rho_a}(a).$$

In particular, $f(x) \ne f(a)$ for any $a \in K$ and any $x \in \mathbf{B}_a(\rho_a)$. As K is compact, we infer that that K is finite. Let $\rho > 0$ be such that $\rho < \rho_a$ for $a \in K$ and

(2)
$$\frac{1}{m_a}\frac{|f(x) - f(a) - \partial f(a)(x-a)|}{|x-a|} \le \frac{\epsilon}{2} \quad \text{whenever } x \in \mathbf{B}_a(\rho).$$

Let $F_\sigma = f^{-1}[\mathbf{B}_b(\sigma)]$ for $0 < \sigma \le s$ and note that F_σ is closed relative to A because f is continuous. Now $\{F_\sigma : 0 < \sigma \le s\}$ is a nested family of closed subsets of the compact set F_s with intersection K. It follows that there is σ such that $0 < \sigma \le s$ and $F_\sigma \subset \cup\{\mathbf{B}_a(\rho) : a \in A\}$. Since $w \in \mathbf{Tan}_b(\mathbf{rng}\, f)$ we may choose $y \in \mathbf{rng}\, f \cap (F_\sigma \sim \{b\})$ such that

$$\left|\frac{1}{|y-b|}(y-b) - w\right| \le \frac{\epsilon}{2}.$$

Let $a \in A$ and $x \in \mathbf{B}_b(\rho_a)$ be such that $y = f(x)$. Then

$$\begin{aligned}\left|w - \partial f(a)\Big(\frac{1}{|y-b|}(x-a)\Big)\right| &= \left|w - \frac{1}{|y-b|}(y-b) + \frac{1}{|f(x)-f(a)|}f(x) - f(a) - \partial f(a)(x-a)\right| \\ &\le \left|w - \frac{1}{|y-b|}(y-b)\right| + \frac{|f(x) - f(a) - \partial f(a)(x-a)|}{|x-a|}\frac{|x-a|}{|f(x)-f(a)|}\right| \\ &\le \epsilon.\end{aligned}$$

□

Theorem. Suppose X and Y are finite dimensional normed spaces, $A \subset X$, $a \in \mathbf{int}\, A$,

$$f : A \to Y$$

and f is continuous at a. Then f is differentiable at a if and only if

$$\mathbf{Tan}_{(a,f(a))}(f)$$

is a linear function from X to Y in which case

$$\mathbf{Tan}_{(a,f(a))}(f) = \partial f(a).$$

Proof. Suppose f is differentiable at a. Let $F(x) = (x, f(x))$ for $x \in A$; note that F is differentiable at a and that $\partial F(a)(v) = (v, \partial f(a))$ whenever $v \in X$.. We may apply the previous Theorem with b and f there replaced by (a,b) and F, respectively, to deduce that $\mathbf{Tan}_{(a,f(a))}(f) = \partial f(a)$.

On the other hand, suppose that $L = \mathbf{Tan}_{(a,f(a))}(f)$ is a linear function from X to Y. Keeping in mind that all norms on a finite dimensional vector space are equivalent, we may suppose $|(x,y)| = |x| + |y|$ for $(x,y) \in X \times Y$. We may suppose without loss of generality that $a = 0$ and $f(a) = 0$.

Let $\epsilon > 0$ and choose $\eta > 0$ such that $\eta(1 + ||L||) < 1$, $\frac{1+(1+||L||)\eta}{1-(1+||L||)\eta} \le 2$ and

$$(1 + ||L||)3\eta \le \epsilon.$$

Choose $\zeta > 0$ such that if $(x, y) \in f \cap \mathbf{B}_0(\zeta)$ then

$$\mathbf{dist}((x, y), L) < \eta|(x, y)|.$$

Finally, using the fact that f is continuous at 0, choose $\delta > 0$ such that if $x \in \mathbf{B}_0(\delta)$ then $x \in A$ and $|(x, f(x))| \le \zeta$.

Suppose $x \in \mathbf{B}_0(\delta)$ and let $y = f(x)$. Then $(x, y) \in f \cap \mathbf{B}_0(\zeta)$ so

$$\mathbf{dist}((x, y), L) < \eta|(x, y)|.$$

We may choose $v \in X$ such that $|(x, y) - (v, L(v))| < \eta(x, y)$ so $|x - v| + |y - L(v)| \le |x| + |y|$. Thus

$$|y| \le |y - L(v)| + |L(v - x)| + |L(x)| \le (1 + ||L||)(|x - v| + |y - L(v)|) + ||L|||x| \le (1 + ||L||)\eta(|x| + |y|) + ||L|||x|$$

so

$$(1 - (1 + ||L||)\eta)|y| \le (1 + (1 + ||L||)\eta)|x|$$

so $|y| \le 2|x|$. It follows that

$$|y - L(x)| \le |y - L(v)| + ||L|||x - v| \le (1 + ||L||)\eta(|x| + |y|) \le (1 + ||L||)3\eta|x| \le \epsilon|x|.$$

Thus f is differentiable at $a = 0$ and its differential is L. □

Theorem. Suppose X is a normed vector space, U is an open subset of X,

$$f : U \to \mathbf{R},$$

$a \in \mathbf{acc}\, A$ and f is differentiable at a.

If $f(x) \le f(a)$ for $x \in A$ then $\partial f(a) \in \mathbf{Nor}_a(A)$.

If $f(x) \ge f(a)$ for $x \in A$ then $-\partial f(a) \in \mathbf{Nor}_a(A)$.

Proof. Exercise. □

Now suppose X is an inner product space. In this case, as we indicated before, we set

$$\mathbf{Nor}_a(A) \;=\; \{w \in X : v \bullet w \le 0 \text{ whenever } v \in \mathbf{Tan}_a(A)\}.$$

Note the the polarity of the inner product carries the present normal cone to the former normal cone.

Definition. The gradient. Suppose $A \subset X$, $f : A \to \mathbf{R}$, and f is differentiable at a. We let

$$\nabla f(a),$$

the **gradient of f at** a, be the counter image of $\partial f(a)$ under the polarity of the inner product; that is, $\nabla f(a)$ is the unique vector in X satisfying

$$\partial f(a)(v) = v \bullet \nabla f(a), \quad v \in X.$$

In this situation the conclusion of the previous Theorem becomes

If $f(x) \le f(a)$ for $x \in A$ then $\nabla f(a) \in \mathbf{Nor}_a(A)$.

If $f(x) \ge f(a)$ for $x \in A$ then $-\nabla f(a) \in \mathbf{Nor}_a(A)$.