1. Sets, relations and functions.

1.1. **Set theory.** We assume the reader is familiar with elementary set theory as it is used in mathematics today. Nonetheless, we shall now give a careful treatment of set theory if only to to allow the reader to become conversant with our notation. Our treatment will be naive and not axiomatic. For an axiomatic treatment of set theory we suggest that the reader consult the Appendix to *General Topology* by J.L. Kelley where one will find a concise and elegant treatment of this subject as well as other references for this subject.

By an **object** we shall mean any thing or entity, concrete or abstract, that might be a part of our discourse. A **set** is a collection of objects and is itself an object. Whenever A is a set and a is one of the objects in the collection A we shall write

$$a \in A$$

and say a **is a member of** A**.** A set is determined by its members; that is, if A and B are sets then

$$A = B \quad \text{if and only if} \quad \text{for every } x,\ x \in A \Leftrightarrow x \in B; \tag{1}$$

this is an axiom; in other words, it is an assumption we make.

The most common way of defining sets is as follows. Suppose $P(x)$ is a formula in the variable x. We will not go into just what this might mean other than to say that (i) if y is a variable then $P(y)$ is a formula and (ii) if a is an object and if each occurrence of x in $P(x)$ is replaced by a then the result $P(a)$ is a statement; we will not go into what this means other than to say that statements are either true of false. At any rate, it is an axiom that there is a set

$$\{x : P(x)\}$$

such that

$$a \in \{x : P(x)\} \quad \text{if and only} \quad P(a) \text{ is true.} \tag{2}$$

That there is only one such set follows from (1). Note that

$$\{x : P(x)\} = \{y : P(y)\}$$

Here is a simple example. Let x be a variable, let

$$P(x) = x \text{ is a cat}$$

and let

$$C = \{x : P(x)\}$$

Then (2) implies that $c \in C$ if and only if c is a cat. One would say that C is the set of all cats. Also, if y is a variable then $C = \{y : P(y)\}$. Of course, $C = \{x : x \text{ is a cat}\}$; that is, we can dispense with P altogether if our goal is only to define the set of cats.

Another way of denoting a set is as follows. Let $a_1, \ldots, a_n$ be a comma delimited (finite!) list of objects. Then

$$\{a_1, \ldots, a_n\}$$

is the set A characterized by the property that $x \in A$ if and only if $x = a_i$ for some $i = 1, \ldots, n$. In particular, for any object a we let **singleton** a equal

$$\{a\}$$

and note that x is a member of singleton a if and only if $x = a$.

We let

$$\emptyset = \{x : x \neq x\}$$

and call this set the **empty set** because it has *no* members. It turns out to be a very convenient abstraction, just like the number 0. (In some treatments of axiomatic set theory it *is* the number 0!) We let

$$\mathcal{U} = \{x : x = x\}$$

and call this set the **universe set** because *every* object is a member of this set. It, like other sets that are large in the sense of being extremely inclusive, causes logical problems like contradictions. We will not worry too much about this. That is what the "naive" in "naive set theory" allows us.

We now present the Russell paradox which is an example of the naivete of naive set theory. Let

$$R = \{x : x \notin x\}.$$

Now either $R \in R$ or $R \notin R$. (No fuzzy logic here!) If $R \in R$ then substitution of R in '$x \notin x$' gives $R \notin R$. On the other hand, if $R \notin R$ then substitution of R in '$x \notin x$' gives a false statement so $R \in R$. We hope to avoid sets like this. Whenever you form sets of grandiose inclusivity you can expect trouble. Whether or not the sets which we will form are sets of grandiose inclusivity depends on whom you talk to. In axiomatic set theory one puts restrictions on sets which prevent the Russell paradox.

1.2. **Set theoretic operations.** Whenever A and B are sets we say A is a **subset** of B and write

$$A \subset B$$

if

$$x \in A \Rightarrow x \in B.$$

Whenever A and B are sets we let

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

and

$$A \sim B = \{x : x \in A \text{ and } x \notin B\};$$

we call these set the **union** of A and B, the **intersection** of A and B and the **complement** of B in A, respectively.

Suppose $\mathcal{A}$ is a set whose members are sets; we shall often call such a set $\mathcal{A}$ a **family** of sets. We let

$$\bigcup \mathcal{A} = \{x : \text{for some } A,\ A \in \mathcal{A} \text{ and } x \in A\}$$

and we let

$$\bigcap \mathcal{A} = \{x : \text{for all } A,\ A \in \mathcal{A} \ \Rightarrow \ x \in A\}.$$

For example, if A and B are sets and $\mathcal{A} = \{A, B\}$ then

$$\bigcup \mathcal{A} = A \cup B \quad \text{and} \quad \bigcap \mathcal{A} = A \cap B.$$

We say $\mathcal{A}$ is **disjointed** if

$$A \cap B = \emptyset \quad \text{whenever } A, B \in \mathcal{A} \text{ and } A \neq B.$$

Note that $\mathcal{A}$ is disjointed iff and only if

$$A, B \in \mathcal{A} \text{ and } A \cap B \neq \emptyset \ \Rightarrow \ A = B.$$

Note that

$$\bigcup \emptyset = \emptyset \quad \text{and that} \quad \bigcap \emptyset = \mathcal{U}.$$

Let X be a set and $\mathcal{A}$ is a nonempty family of subsets of X. We leave to the reader as an exercise the verification of the **DeMorgan laws**:

$$X \sim \cup\mathcal{A} = \cap\{X \sim A : A \in \mathcal{A}\} \quad \text{and} \quad X \sim \cap\mathcal{A} = \cup\{X \sim A : A \in \mathcal{A}\}.$$

Wait a minute! We have not said what we mean by

$$\{X \sim A : A \in \mathcal{A}\};$$

it should really be

$$\{B : \text{for some } A,\ A \in \mathcal{A} \text{ and } B = X \sim A\}$$

we shall abuse notation in this manner unless it might cause confusion. We also point out that the proof of the DeMorgan Laws reduces immediately to rules of elementary logic. *We assume the reader is quite proficient at elementary logic.* Heh, heh.

We let

$$2^X = \{A : A \subset X\}$$

and call this set the **power set of** X.

We $\mathcal{A}$ is a **partition** of X if $\mathcal{A}$ is a disjointed family of sets such that

$$X = \bigcup \mathcal{A}.$$

A family $\mathcal{C}$ of sets is **nested** if either $C \subset D$ or $D \subset C$ whenever $C, D \in \mathcal{C}$.

1.3. **Ordered pairs and relations.** Suppose a and b are objects. We let

$$(a, b) = \{\{a\}\} \cup \{\{a, b\}\}$$

and note that

$$\cup(a, b) = \{a\} \cup \{b\} \quad \text{and} \quad \cap(a, b) = \{a\}. \tag{1}$$

Proposition 1.1. Suppose a, b, c and d are objects. Then

$$(a, b) = (c, d) \ \Leftrightarrow \ a = c \text{ and } b = d.$$

Proof. Use (1). □

We say p is an **ordered pair** if there exist objects a and b such that

$$p = (a, b).$$

It follows from the preceding Proposition that if a and b are uniquely determined so we may define the **first coordinate** of p to be a and the **second coordinate** of (a, b) to be b.

A **relation** is a set whose members are ordered pairs. Whenever r is a relation we let

$$\mathbf{dmn}\, r = \{x : \text{ for some } y,\ (x, y) \in r\}$$

and we let

$$\mathbf{rng}\, r = \{y : \text{ for some } x,\ (x, y) \in r\};$$

we call these sets the **domain** and **range** of r, respectively; we let

$$r^{-1} = \{(x, y) : (y, x) \in r\}$$

and call this relation the **inverse** of r. If r and s are relations we let

$$r \circ s = \{(x, z) : \text{ for some } y,\ (x, y) \in s \text{ and } (y, z) \in r\};$$

we call this relation the **composition** of the relations s and r. We could be *very* formal and write

$$r \circ s = \{(x, z) : \exists y(((x, y) \in s) \wedge ((y, z) \in t))\}.$$

Exercise 1.1. Show that composition of relations is associative.

Well, maybe I'll do this one for you.

Proof. The following statements are equivalent. (Make sure you see why!)

(1) $$(x, w) \in r \circ (s \circ t)$$

(2) $$\exists z(((x, z) \in s \circ t) \wedge ((z, w) \in r))$$

(3) $$\exists z((\exists y(((x, y) \in t) \wedge (y, z) \in s)) \wedge ((z, w) \in r))$$

(4) $$\exists z(\exists y(((x, y) \in t) \wedge (y, z) \in s) \wedge ((z, w) \in r))$$

(5) $$\exists z(\exists y(((y, z) \in s) \wedge ((z, w) \in r) \wedge ((x, y) \in t)))$$

(6) $$\exists y(\exists z(((y, z) \in s) \wedge ((z, w) \in r) \wedge ((x, y) \in t)))$$

(7) $$\exists y(\exists z(((y, z) \in s) \wedge ((z, w) \in r)) \wedge ((x, y) \in t)))$$

(8) $$\exists y(((y, w) \in r \circ s) \wedge ((x, y) \in t))$$

(9) $$(x, w) \in (r \circ s) \circ t$$

□

Exercise 1.2. Show that $(r \circ s)^{-1} = s^{-1} \circ r^{-1}$ whenever r and s are relations.

Definition 1.1. Suppose r is a relation and A is a set. We let

$$r|A = \{(x, y) : (x, y) \in r \text{ and } x \in A\}$$

and call this relation r **restricted to** A and we let

$$r[A] = \{y : \text{ for some } x,\, x \in A \text{ and } (x, y) \in r\}$$

and call this set r **of** A.

Exercise 1.3. Show that

$$r[s[A]] = (r \circ s)[A]$$

whenever r and s are relations and A is a set.

Proposition 1.2. Suppose r is a relation and $\mathcal{A}$ is a family of sets. Then

$$r[\bigcup \mathcal{A}] = \bigcup \{r[A] : A \in \mathcal{A}\}.$$

Proof. Suppose $y \in r[\bigcup \mathcal{A}]$. Then there is $x \in \bigcup \mathcal{A}$ such that $(x, y) \in r$. Since $x \in \bigcup \mathcal{A}$ there is $A \in \mathcal{A}$ such that $x \in A$ so $y \in r[A]$ so $y \in \bigcup\{r[A] : A \in \mathcal{A}\}$.

On the other hand, suppose $y \in \bigcup\{r[A] : A \in \mathcal{A}\}$. There is $A \in \mathcal{A}$ such that $y \in r[A]$. In particular, there is $x \in A$ such that $(x, y) \in r$ so, as $x \in \bigcup \mathcal{A}$, $y \in r[\bigcup \mathcal{A}]$. □

Remark 1.1. We will give an example shortly of a relation r and a nonempty family of sets $\mathcal{A}$ such that $r[\bigcap \mathcal{A}] \neq \bigcap\{r[A] : A \in \mathcal{A}\}$.

Definition 1.2. Whenever A and B are sets we let

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

and call this set the **Cartesian product of** A **and** B. If A is a set we say r is a **relation on** A if $r \subset A \times A$ and we frequently write

$$x \ r \ y \quad \text{instead of} \quad (x, y) \in r.$$

1.4. **Functions.** We say a relation f is a **function** if

$$(x, y_1) \in f \text{ and } (x, y_2) \in f \Rightarrow \ y_1 = y_2.$$

In other words, $f[\{x\}]$ has at most one member for any object x. Whenever f is a function and $x \in \mathbf{dmn}\, f$ we let

$$f(x) \quad \text{or} \quad f_x$$

be the unique member of $f[\{x\}]$ and call this object the **image of** x **under** f. We write

$$f : X \to Y$$

and say f **is a function from** X **to** Y if

(i) f is a function;
(ii) $X = \mathbf{dmn}\, f$; and
(iii) $\mathbf{rng}\, f \subset Y$.

Note that if $\mathbf{rng}\, f \subset Y_i$, $i = 1, 2$, then

$$f : X \to Y_i, \quad i = 1, 2.$$

We say the function f is **univalent** if

$$(x_1, y) \in f \text{ and } (x_2, y) \in f \Rightarrow \ x_1 = x_2;$$

this amounts to saying that f^{-1} is a function which we call the **inverse function to** f. Note that if f and g are functions then $g \circ f$ is a function whose domain is $f^{-1}[\mathbf{dmn}\, g]$ and whose range is $g[\mathbf{rng}\, f]$.

Proposition 1.3. Suppose f is a function and B is a set. Then

$$x \in f^{-1}[B] \;\Leftrightarrow\; f(x) \in B.$$

Proof. We have $x \in f^{-1}[B]$ iff for some y, $y \in B$ and $(y, x) \in f^{-1}$ iff for some y, $y \in B$ and $(x, y) \in f$ iff for some y, $y \in B$ and $y = f(x)$ iff $f(x) \in B$. □

Here is an extremely useful fact about functions.

Proposition 1.4. Suppose $\mathcal{A}$ is a family of sets. Then

$$f^{-1}[\bigcap \mathcal{A}] = \bigcap\{f^{-1}[A] : A \in \mathcal{A}\} \quad \text{and} \quad f^{-1}[\bigcup \mathcal{A}] = \bigcup\{f^{-1}[A] : A \in \mathcal{A}\}.$$

Suppose A and B are sets. Then

$$f^{-1}[A \sim B] = f^{-1}[A] \sim f^{-1}[B].$$

Proof. Exercise for the reader. Note that

$$f^{-1}[\bigcup \mathcal{A}] = \bigcup\{f^{-1}[A] : A \in \mathcal{A}\}$$

is a special case of a previous Proposition. □

Remark 1.2. Keeping in mind that forward images of unions are preserved by relations we see that this Proposition says that all the set theoretic operations are preserved by taking the counterimage under a function.

That forward images of unions are preserved under functions follows from earlier work. This is *not* true for forward images of intersections as the following simple example illustrates.

Example 1.1. Let $f = \{(0, 0)\} \cup \{(1, 0)\}$, let $A = \{0\}$ and let $B = \{1\}$. Then

$$f[A \cap B] = f[\emptyset] = \emptyset \neq \{0\} = f[A] \cap f[B].$$

Exercise 1.4. Prove or disprove:

$$f[A \sim B] = f[A] \sim f[B]$$

whenever f is a function and A and B are sets.

1.5. Equivalence relations.

Definition 1.3. Suppose r is a relation on the set X. We say r is **reflexive** if

$$a \in X \Rightarrow (a, a) \in r;$$

we say r is **strict** if

$$a \in X \Rightarrow (a, a) \notin r;$$

we say r is **symmetric** if

$$(a, b) \in r \Rightarrow (b, a) \in r;$$

we say r is **transitive** if

$$(a, b) \in r \text{ and } (b, c) \in r \Rightarrow (a, c) \in r;$$

we say r is **trichotomous** if for each $(a, b) \in X \times X$ exactly one of the following holds:

$$(a, b) \in r; \quad a = b; \quad (b, a) \in r.$$

Definition 1.4. We say r is an **equivalence relation on** X if it is reflexive, symmetric and transitive in which case we let

$$X/r = \{r[\{x\}] : x \in X\}.$$

Here is a basic theorem about equivalence relations that is used throughout pure mathematics in building new mathematical objects out of old ones.

Theorem 1.1. Suppose r is an equivalence relation on X. Then X/r is a partition of X each of whose members is nonempty.

On the other hand, if $\mathcal{A}$ is a partition of X and no member of $\mathcal{A}$ is empty then

$$r = \cup\{A \times A : A \in \mathcal{A}\}$$

is an equivalence relation on X and $X/r = \mathcal{A}$.

Exercise 1.5. We leave the proof of this Theorem as an exercise for the reader.

2. Partial orders.

Definition 2.1. We say the relation p **partially orders** the set X if p is strict and transitive. When this is the case we often use $<$ to stand for p and we often write

$$x < y \quad \text{instead of} \quad (x, y) \in p$$

as well as

$$\begin{aligned} x \leq y &\quad \text{for} \quad x < y \text{ or } x = y; \\ x > y &\quad \text{for} \quad y < x; \\ x \geq y &\quad \text{for} \quad x > y \text{ or } x = y \end{aligned}$$

whenever $x, y \in X$.

Proposition 2.1. Suppose $<$ partially orders the set X and $x, y \in X$. Then at most one of the following holds:

$$x < y; \quad x = y; \quad y < x;$$

moreover,

$$x \leq y \text{ and } y \leq x \;\Leftrightarrow\; x = y.$$

Proof. Suppose $x \neq y$. Were it the case that $x < y$ and $y < x$ we would have $x < x$ by transitivity but this is excluded because a partial order is strict. Thus the first assertion holds. The second is a purely logical consequence of the first. □

Proposition 2.2. Suppose X is a set, p partially orders X and

$$\prec$$

equals

$$\{(U, V) : U \subset V \text{ and } U \neq V\}.$$

Then $\prec$ is partial ordering of 2^X.

Moreover, if

$$f = \{(x, \{w \in X : w \leq x\}\}) \subset X \times 2^X$$

then

$$f : X \to 2^X,$$

f is univalent and

$$x, y \in X \text{ and } x < y \;\Rightarrow\; f(x) \prec f(y).$$

Proof. We leave it to the reader to supply simple proofs that $\prec$ partially orders 2^X and that $f : X \to 2^X$.

In what follows we will use Proposition 2.1 repeatedly.

Suppose $x, y \in X$. If $f(x) = f(y)$ then $x \in f(x) = f(y)$ so $x \leq y$ and $y \in f(y) = f(x)$ so $y \leq x$ which gives $x = y$; thus f is univalent.

Suppose $x, y \in X$ and $x < y$. If $w \in f(x)$ then $w \leq x$ which by transitivity implies $w < y$ so $w \in f(y)$ which is to say that $f(x) \subset f(y)$. Were it the case that $y \in f(x)$ we would have $y \leq x$ which is incompatible with $x < y$; since $x \in f(x)$ we have $f(x) \neq f(y)$. Thus $f(x) \prec f(y)$. □

Remark 2.1. The Proposition says that $\prec$ is the mother of all partial orderings on X.

Definition 2.2. Suppose X is a set, $<$ partially orders X and and $A \subset X$.

We say a u is an **upper bound for** A if $u \in X$ and

$$a \in A \;\Rightarrow\; a \leq u.$$

We say g is a **greatest member of** A if $g \in A$ and g is an upper bound for A.

We say M is a **maximal element of** A if $M \in A$ and

$$M < a \text{ for no } a \in A.$$

We say l is an **lower bound for** A if $l \in X$ and

$$a \in A \;\Rightarrow\; l \leq a.$$

We say l is a **least element of** A if $l \in A$ and l is a lower bound for A.

We say m is a **minimal element of** A if $m \in A$ and

$$a < m \text{ for no } a \in A.$$

Proposition 2.3. Suppose X is a set and $<$ partially orders X. Then

(i) a greatest member of A is a maximal member of A;
(ii) A has at most one greatest member;
(iii) a least member of A is a minimal member of A;
(iv) A has at most one least member.

Proof. Straightforward exercise for the reader. □

Definition 2.3. Suppose X is a partially ordered set and $A \subset X$.

A least member of the set of upper bounds for A is called a **least upper bound for** A.

By Proposition 2.3 any least upper bound for A is unique; it called **the supremum of** A and it is denoted

$$\sup A \quad \text{or} \quad \mathbf{l.u.b.}\, A.$$

A greatest member of the set of lower bounds for A is called a **greatest lower bound for** A**.** By Proposition 2.3 any greatest lower bound for A is unique; it called **the infimum of** A and is denoted

$$\inf A \quad \text{or} \quad \mathbf{g.l.b.}\, A.$$

Proposition 2.4. Suppose X is a partially ordered set. The following two conditions are equivalent.

(i) If A is a nonempty subset of X and there is an upper bound for A then there is a least upper bound for A.

(ii) If A is a nonempty subset of X and there is a lower bound for A then there is a greatest lower bound for A.

Proof. For any subset A of X let $U(A)$ be the set of upper bounds for A and let $L(A)$ be the set of lower bounds for A.

Suppose (i) holds, $A \subset X$ and $A \neq \emptyset$ and $L(A) \neq \emptyset$. Note that $A \subset U(L(A))$. Thus $L(A)$ and $U(L(A))$ are nonempty so $U(L(A))$ has a least member b. I claim that b is a greatest lower bound for A.

Since $A \subset U(L(A))$ we have $b \leq a$ for $a \in A$ so $b \in L(A)$. Suppose $c \in L(A)$. Since $b \in U(L(A))$ we have $c \leq b$. That is, b is a greatest lower bound for A.

In a similar fashion one shows that (ii) implies (i). □

Definition 2.4. We say $<$ is **complete** if either of the equivalent conditions in the above Proposition holds.

Example 2.1. Suppose X is a set and $\prec$ is the partial order on 2^X defined in Proposition 2.2.

Let $\mathcal{A}$ be a nonempty family of subsets of X. That is, $\mathcal{A} \subset 2^X$. I claim that $\cup\mathcal{A}$ is a least upper bound for $\mathcal{A}$ (with respect to $\prec$).

Suppose $A \in \mathcal{A}$. If $x \in A$ then $x \in \cup\mathcal{A}$ so $A \subset \cup\mathcal{A}$. Thus $\cup\mathcal{A}$ is an upper bound for $\mathcal{A}$.

Suppose the subset B of X is an upper bound for $\mathcal{A}$. If $x \in \cup\mathcal{A}$ then there is $A \in \mathcal{A}$ such that $x \in A$. Since B is an upper bound for $\mathcal{A}$ we have $A \subset B$. Thus $x \in B$ and we have show that $\cup\mathcal{A} \subset B$.

This verifies my claim.

Exercise 2.1. Show that $\cap\mathcal{A}$ is a greatest lower bound for $\mathcal{A}$.

Definition 2.5. Let X be a set. We say l **linearly orders** X if l partially orders X and l is trichotomous. We say w **well orders** X if w linearly orders X and every nonempty subset of X has a least element.

Definition 2.6. Suppose $<$ linearly orders X and $I \subset X$. Then I is an **interval** if

$$x, z \in I,\ y \in X \text{ and } x < y < z \ \Rightarrow\ y \in I.$$

Proposition 2.5. Suppose X is a set, $<$ linearly orders X and $A \subset X$. Then

(i) the set of maximal members of A has at most one member;
(ii) the set of minimal members of A has at most one member;

Proof. Straightforward exercise for the reader. □

2.1. **Equipotence and cardinal numbers.** Suppose X and Y are sets. We say that X is **equipotent with** Y and write

$$X \approx Y$$

if there exists a relation f such that $f : X \to Y$ and $f^{-1} : Y \to X$. It is obvious that

$$X \approx X$$

whenever X is a set;

$$X \approx Y \ \Rightarrow Y \approx X$$

whenever X and Y are sets and

$$X \approx Y \text{ and } Y \approx Z \ \Rightarrow\ X \approx Z$$

whenever X, Y and Z are sets. Thus we have introduced an equivalence relation on the set of all sets the equivalence classes corresponding to which are called **cardinal numbers**. (Forming the set of all sets was never allowed in public, was allowed in secret until about forty years ago and is forbidden under any circumstances today!)

Theorem 2.1. Suppose X is nonempty set and

$$f : X \to 2^X.$$

Then

$$\mathbf{rng}\, f \neq 2^X.$$

Remark 2.2. This simple but fundamental Theorem says that 2^X is larger than X when X is nonempty. The proof is an abstraction (the right one, in my opinion) of what is called Cantor's diagonal argument.

Proof. Let $A = \{x \in X : x \notin f(x)\}$. Were it the case that $A \in \mathbf{rng}\, f$ there would be $a \in X$ such that $f(a) = A$. But then

$$a \in A \;\Rightarrow\; a \notin f(a) \;\Rightarrow a \notin A$$

and

$$a \notin A \;\Rightarrow\; a \in f(a) \;\Rightarrow a \in A$$

neither of which is possible. Thus $A \notin \mathbf{rng}\, f$. □