1. The real numbers.

1.1. **Ordered rings.**

Definition 1.1. By an **ordered commutative ring with unity** we mean an ordered sextuple

$$(R, +, 0, *, 1, P)$$

such that $(R, +, 0, *, 1)$ is a commutative ring with unity and such that P is a subset of R with the following properties:

(i) if $x \in R$ the exactly one of the following holds:

$$-x \in P; \quad x = 0; \quad x \in P;$$

(ii) if $x \in P$ and $y \in P$ then $x + y \in P$;
(iii) if $x \in P$ and $y \in P$ then $xy \in P$.

If $x \in P$ we say x **is positive** and if $-x \in P$ we say x **is negative**.

We will frequently say "R is an ordered ring with positive elements P" instead of "$(R, \alpha, 0, \mu, 1, P)$ is an ordered ring".

Suppose R is an ordered ring with positive elements P.

Proposition 1.1. Suppose $x, y \in R$ and $xy = 0$. Then either $x = 0$ or $y = 0$.

Remark 1.1. That is, R is an **integral domain**.

Proof. We have

$$\begin{aligned} x \in P \text{ and } y \in P \ &\Rightarrow\ xy \in P \ \Rightarrow\ xy \neq 0; \\ x \in P \text{ and } -y \in P \ &\Rightarrow\ -(xy) = x(-y) \in P \ \Rightarrow\ xy \neq 0; \\ -x \in P \text{ and } y \in P \ &\Rightarrow\ -(xy) = (-x)y \in P \ \Rightarrow\ xy \neq 0; \\ -x \in P \text{ and } -y \in P \ &\Rightarrow\ xy = (-x)(-y) \in P \ \Rightarrow\ xy \neq 0. \end{aligned}$$

Thus $x = 0$ or $y = 0$. □

Definition 1.2. We let

$$<$$

be the set of $(x, y) \in R \times R$ such that $y - x \in P$.

Proposition 1.2. $<$ linearly orders R and $P = \{x \in R : x > 0\}$.

Remark 1.2. In particular, if $a \in R$ then exactly one of the following holds:

$$-a > 0; \quad a = 0; \quad a > 0.$$

(Don't forget our convention about $\leq, >$, etc.)

Proof. We leave this as a straightforward exercise for the reader. □

Theorem 1.1. Suppose $a, b, c \in R$. Then

(i) $a > 0 \Leftrightarrow -a < 0$;
(ii) $a < 0 \Leftrightarrow -a > 0$;
(iii) $a \neq 0 \Rightarrow a^2 > 0$;
(iv) $1 > 0$;
(v) $a < b \Leftrightarrow a + c < b + c$;

(vi) $a < b$ and $c > 0 \Rightarrow ac < bc$;

Proof. Exercise for the reader. □

1.1. Absolute values. For $a \in R$ we set

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{else} \end{cases}$$

and note that

$$|a| + a \geq 0 \quad \text{and} \quad |a| - a \geq 0.$$

Note that

$$|a| \leq b \Leftrightarrow a \leq b \quad \text{and} \quad -a \leq b \tag{1}$$

whenever $a, b \in R$.

We can now easily prove that

$$|a + b| \leq |a| + |b|, \ a, b \in R. \tag{2}$$

Indeed this inequality is equivalent to the inequalities

$$a + b \leq |a| + |b| \quad \text{and} \quad -a - b \leq |a| + |b|.$$

The first of these is equivalent to $0 \leq (|a| - a) + (|b| - b)$ and the second is equivalent to $0 \leq (|a| + a) + (|b| + b)$ each of which follow from (1).

We have

$$||a| - |b|| \leq |a - b|, \ a, b \in \mathbb{R}. \tag{3}$$

Indeed this inequality is equivalent to the inequalities

$$|a| - |b| \leq |a - b| \quad \text{and} \quad -|b| + |a| \leq |a + b|$$

by (1). These inequalities in turn are equivalent to the inequalities

$$|a| \leq |a - b| + |b| \quad \text{and} \quad |a| \leq |a + b| + |b|$$

each of which follows from (2).

Theorem 1.2. There is one any only one function

$$f : \mathbb{Z} \to R$$

such that

(1) $\mathbf{rng}\, f \neq \{0\}$ and

(2) $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ whenever $x, y \in \mathbb{Z}$.

Moreover,

(3) $f(x) < f(y)$ whenever $x, y \in \mathbb{Z}$ and $x < y$.

Remark 1.3. We will frequently not distinguish between the integer n and its image $f(n)$ in R.

Remark 1.4. By virtue of this Theorem, the ring integers modulo a nonzero integer m cannot be ordered.

Proof. By induction, there is one and only function $f^+ : \mathbb{N} \to R$ such that $f^+(0) = 0$ and $f^+(n+1) = f^+(n) + 1$ whenever $n \in \mathbb{N}$. Using induction and the fact that the sum of positive members of R is positive we infer that $f^+(n) > 0$ whenever $n \in \mathbb{N}^+$. Using induction again one shows that f^+ preserves arithmetic operations and order.

Now let

$$f = \{(m - n, f^+(m) - f^+(n)) : (m, n) \in \mathbb{N} \times \mathbb{N}\}.$$

We leave it to the reader to verify that f is a function with the desired properties. □

Definition 1.3. An ordered field is an ordered ring which is a field.

Proposition 1.3. If F is the field of quotients of R then

$$\left\{\frac{x}{y} : x, y \in P\right\}$$

is a set of positive elements for an ordered field structure on F.

Proof. We leave this as a straightforward exercise for the reader. □

Now suppose F is an ordered field.

Theorem 1.3. There is one any only one function

$$g : \mathbb{Q} \to F$$

such that

(1) $\mathbf{rng}\, g \neq \{0\}$ and

(2) $g(x + y) = g(x) + g(y)$ and $g(xy) = g(x)g(y)$ whenever $x, y \in \mathbb{Z}$.

Moreover,

(3) $g(x) < g(y)$ whenever $x, y \in \mathbb{Q}$ and $x < y$.

Remark 1.5. We will frequently not distinguish between the rational number q and its image $f(q)$ in F.

Proof. Let $f : \mathbb{Z} \to F$ be such that $\mathbf{rng}\, f \neq \{0\}$ and $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ whenever $x, y \in \mathbb{Z}$. Let

$$g = \left\{\left(\frac{x}{y}, \frac{f(x)}{f(y)}\right) : (x, y) \in \mathbb{Z} \times (\mathbb{Z} \sim \{0\})\right\}.$$

We leave it to the reader to verify that g is a function which has the desired properties. □

Theorem 1.4. Suppose $a \in F$. Then

$$a > 0 \Leftrightarrow \frac{1}{a} > 0.$$

Proof. Exercise for the reader. □

Theorem 1.5. Suppose $a, b \in F$ and

$$a \leq b + \epsilon \quad \text{for all } \epsilon > 0.$$

Then

$$a \leq b.$$

Remark 1.6. This need *not* be true over an ordered ring. For example $1 \leq 0 + \epsilon$ for every positive integer ϵ.

Proof. Suppose, contrary to the Theorem, $b < a$. Let $\epsilon = (a - b)/2 > 0$. Then

$$a - (b + \epsilon) = a - \left(b + \frac{a-b}{2}\right) = \frac{a-b}{2} > 0$$

so $b + \epsilon < a$ which is a contradiction. □

Proposition 1.4. The the following are equivalent:

(i) for each $R \in F$ with $R > 0$ there is $N \in \mathbb{N}^+$ such that $R < N$;
(ii) for each $\epsilon \in F$ with $\epsilon > 0$ there is $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$.

Proof. Suppose (i) holds and $\epsilon > 0$. There is then $N \in \mathbb{N}^+$ such that $\frac{1}{\epsilon} < N$ so $\frac{1}{N} < \epsilon$ so (ii) holds. In like manner one shows that (ii) implies (i). □

Definition 1.4. We say an ordered field is **Archimedean** if either of the equivalent conditions in the previous Proposition hold.

Theorem 1.6. Suppose F is an Archimedean ordered field.

(i) Whenever $c, \epsilon \in F$ and $\epsilon > 0$ there exists a unique integer m such that $m\epsilon \leq c < (m+1)\epsilon$.
(ii) Whenever $a, b \in F$ and $a < b$ there is a rational number r such that $a < r < b$.

Proof. To prove (i) we suppose $c \geq 0$ and leave the case $c < 0$ to the reader. The set $A = \{n \in \mathbb{N}^+ : c/\epsilon < n\}$ is nonempty and therefore has a least element l. Let $m = l - 1$. Evidently, $c/\epsilon < l = m + 1$ so $c < (m+1)\epsilon$. Since l is the least element of A we have $m = l - 1 \notin A$ which amounts to $c/\epsilon \geq m$ so $m\epsilon \leq c$.

To prove (ii), choose a positive integer n such that $1/n < b - a$ and then use (i) to choose an integer m such that $m/n \leq a < (m+1)/n$. Then

$$b = a + (b - a) > a + 1/n \geq m/n + 1/n = (m+1)/n$$

so we may take $r = (m+1)/n$. □

Example 1.1. The field of rational functions over $\mathbb{Q}$ which is, by definition, the field of quotients of the polynomials over $\mathbb{Q}$, is an ordered field (Where did the order come from?) which is *not* Archimedian.

There is also the subject of *nonstandard analysis* where one tries to make a field with infinitesimals in it. We'll stick to standard analysis.

1.2. **Completely ordered fields.**

Definition 1.5. By a **completely ordered field** we mean an ordered field whose ordering is complete.

Theorem 1.7. Any completely ordered field is Archimedean.

Proof. Suppose F is a completely ordered field, $R \in F$ and $R > 0$. If there is no positive integer N such that $R < N$ then R is an upper bound for the positive integers which must therefore have a least upper bound L. Then would then be a positive integer n such that $L-1 < n$ which implies $L < n+1$ which is incompatible with L being an upper bound for the set of positive integers. □

Theorem 1.8. Suppose F is a completely ordered field and $x \in F$. Then

$$x = \sup\{r \in \mathbb{Q} : r < x\}.$$

Proof. Let $X = \{r \in \mathbb{Q} : r < x\}$. By Theorem 1.6(ii) there is $q \in \mathbb{Q}$ such that $x - 1 < q < x$ so X is not empty. Since x is an upper bound for X there is a least upper bound w for X. Since x is an upper bound for X we have $w \leq x$. Were it the case that $w < x$ we could Theorem 1.6(ii) to obtain a rational number q such that $w < q < x$. As $q \in X$ this is incompatible with w being an upper bound for X; so $w = \sup X = x$. □

Theorem 1.9. The uniqueness of completely ordered fields. Suppose F_i, $i = 1, 2$, are completely ordered fields. There is one and only function

$$f : F_1 \to F_2$$

such that

(i) $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ whenever $x, y \in F_1$ and
(ii) $f(x) < f(y)$ whenever $x, y \in F_1$ and $x < y$.

Moreover,

$$f(\sup A) = \sup f[A]$$

for any nonempty subset A of X_1 which has an upper bound.

Proof. For $x \in F_1$ let $f(x)$ be the supremum in F_2 of $\{r \in \mathbf{Q} : r < x\}$. We leave it as a straightforward but tedious exercise to verify f has the desired properties. Its uniqueness is evident. □

Exercise 1.1. Suppose F is a completely ordered field, n is a positive integer, $x \in F$ and $x > 0$. As an exercise you will show that there is $w \in F$ such that $w > 0$ and $w^n = x$. Let

$$W = \{v \in F : v \geq 0 \text{ and } v^n < x\}.$$

Note that $0 \in W$ so $W \neq \emptyset$. Show that W has an upper bound so that $w = \sup W$ exists. Show that $w^n = x$.

Let me get you started. Suppose $w^n \neq x$. If $w^n < x$ come up with $\epsilon > 0$ such that $(w + \epsilon)^n < x$ thereby obtaining a contradiction; in a similar fashion, if $w^n > x$

come up with $\epsilon > 0$ such that $(w-\epsilon)^n > x$. You will need the binomial theorem. You might try the case $n = 2$ first.

So if there is a completely ordered field it is unique structurally. But is there one? In the next two subsections we will give two rather different constructions of a completely ordered field.

Theorem 1.10. Suppose F is an ordered field. Show that the following are equivalent:

(i) F is complete.

(ii) If I is an interval then one of the following holds:

- (a) $I = \emptyset$;
- (b) $I = \{d\}$ for some $d \in F$;
- (c) $I = (a, b)$ for some $a, b \in F$ with $a < b$;
- (d) $I = [a, b)$ for some $a, b \in F$ with $a < b$;
- (e) $I = (a, b]$ for some $a, b \in F$ with $a < b$;
- (f) $I = [a, b]$ for some $a, b \in F$ with $a < b$;
- (g) $I = (-\infty, c)$ for some $c \in F$;
- (h) $I = (-\infty, c]$ for some $c \in F$;
- (i) $I = (c, -\infty)$ for some $c \in F$;
- (j) $I = [c, \infty)$ for some $c \in F$;
- (k) $I = F$.

Proof. This follows readily from the Exercise in which you were asked to show that F was complete if and only if the only initial segments are

$$\{w \in F : w < x\} \quad \text{and} \quad \{w \in F : w \leq x\}$$

corresponding to $x \in F$. □

1.3. **Dedekind cuts.**

Definition 1.6. Dedekind Cuts. We say as subset D of $\mathbb{Q}$ is a **Dedekind cut** if D is an initial segment and D has no largest element.

We let

$$\overline{\mathbb{R}}$$

be the family of Dedekind cuts and and we call the members of $\overline{\mathbb{R}}$ **extended real numbers**.

The following simple Proposition will allow us to define a natural linear ordering on $\overline{\mathbb{R}}$.

Proposition 1.5. Suppose $D, E \in \overline{\mathbb{R}}$ and $E \sim D \neq \emptyset$. Then $D \subset E$.

Proof. Suppose $p \in D$ Choose a member q of $E \sim D$. Were it the case that $q < p$ we would have $q \in D$ since D is a Dedekind cut. Thus $p < q$ so $p \in E$ since E is a Dedekind cut. □

Suppose $D, E \in \overline{\mathbb{R}}$. We declare

$$D < E \quad \text{if } D \neq E \text{ and } D \subset E.$$

It is obvious that $<$ is partial ordering of $\overline{\mathbb{R}}$ and the previous Proposition implies $<$ is trichotomous so $<$ is a linear ordering of $\overline{\mathbb{R}}$. We let

$$-\infty = \emptyset \quad \text{and we let} \quad \infty = \mathbb{Q}.$$

Evidently,

$$-\infty, \infty \in \overline{\mathbb{R}} \quad \text{and} \quad -\infty \leq D \leq \infty \quad \text{whenever } D \in \overline{\mathbb{R}}.$$

In particular, $-\infty$ is a lower bound and ∞ is an upper bound for any subset of $\overline{\mathbb{R}}$.

Proposition 1.6. $<$ is a complete linear ordering of $\overline{\mathbb{R}}$.

Proof. Suppose $\mathcal{A}$ is a nonempty subfamily of $\overline{\mathbb{R}}$. We leave it as a straight forward exercise for the reader to verify that $\cup\mathcal{A}$ is a Dedekind cut. If $A \in \mathcal{A}$ then $A \subset \cup\mathcal{A}$ so $\cup\mathcal{A}$ is an upper bound for $\mathcal{A}$. If $E \in \overline{\mathbb{R}}$ and E is an upper bound for $\mathcal{A}$ then $A \subset E$ whenever $A \in \mathcal{A}$ so $\cup\mathcal{A} \subset E$. Thus $\cup\mathcal{A}$ is the least upper bound for $\mathcal{A}$. □

If $A \subset \overline{\mathbb{R}}$ we let

$$\inf A \quad \text{and} \quad \sup A$$

be the greatest lower bound or *infimum of* A and least upper bound or *supremum of* A, respectively. Note that

$$\inf \emptyset = \infty \quad \text{and} \quad \sup \emptyset = -\infty.$$

It is a generally a good idea to exclude the possibility of $\inf \emptyset$ or $\sup \emptyset$ occurring in an argument.

Recall that

$$\mathbb{I}(q) = \{p \in \mathbb{Q} : p < q\}.$$

We leave it as a straight forward exercise for the reader to verify that $\mathbb{Q} \ni r \mapsto \mathbb{I}(q)$ carries $\mathbb{Q}$ univalently into $\overline{\mathbb{R}}$ and that $\mathbb{I}(p) < \mathbb{I}(q)$ if $p, q \in \mathbb{Q}$ and $p < q$.

We let

$$\mathbb{R} = \overline{\mathbb{R}} \sim \{-\infty, \infty\}$$

and we call the members of $\mathbb{R}$ **real numbers**. Note that if $D \in \overline{\mathbb{R}}$ then $D \in \mathbb{R}$ if and only if D is nonempty and has an upper bound in $\mathbb{Q}$.

We define a field structure on $\mathbb{R}$ as follows. Suppose $D, E \in \mathbb{R}$. We let

$$D + E = \{d + e : d \in D \text{ and } e \in E\}.$$

If $0 \in D \cap E$ we let

$$DE = \{p \in \mathbb{Q} : \text{for some } q, r,\ q \in D,\ r \in E,\ q > 0,\ r > 0 \text{ and } p < qr\}.$$

We leave it to the reader to modify this definition of multiplication appropriately to treat the case when $0 \notin D \cap E$. It is then a straightforward but rather tedious exercise to verify that the ordered field axioms are satisfied where the additive neutral element is $\mathbb{I}(0)$ and the multiplicative neutral element is $\mathbb{I}(1)$. It is probably a waste of time to do this exercise. See what follows for an alternate construction of the real number systems in which the field properties directly result.

We extend $+$ and $*$ as follows: For any $a \in \mathbb{R}$ we set

$$\pm\infty + a = \pm\infty = a + \pm\infty$$
$$a(\pm\infty) = \pm\infty = (\pm\infty)a \quad \text{if } a > 0$$
$$a(\pm\infty) = \mp\infty = (\pm\infty)a \quad \text{if } -a < 0$$

and we set

$$0(\pm\infty) = 0 = (\pm\infty)0 \quad \text{if } -\infty < a < 0$$

All other arithmetic operations involving $\pm\infty$ are undefined.

The following summarizes some absolutely fundamental properties of inf and sup.

Theorem 1.11. Suppose A and B are nonempty subsets of $\overline{\mathbb{R}}$. Then

(i) $\inf A \le \sup A$;
(ii) if $A \subset B$ then $\inf B \le \inf A$ and $\sup A \le \sup B$;
(iii) $\sup A \le \inf B$ if and only if $a \le b$ whenever $a \in A$ and $b \in B$;
(iv) if $b \in \mathbb{R}$ then $\inf A \le b$ if and only if for each $\epsilon > 0$ there is $a \in A$ such that $a \le b + \epsilon$;
(v) $b \le \inf A$ if and only if $b \le a$ for all $a \in A$;
(vi) $b \le \sup A$ if and only if for each $\epsilon > 0$ there is $a \in A$ such that $b \le a + \epsilon$;
(vii) $\sup A \le b$ if and only if $a \le b$ for all $a \in A$.

Proof. Exercise for the reader. □

Theorem 1.12. Suppose $a \in \mathbb{R}$ and B is a nonempty subset of $\mathbb{R}$. Then

(i) $\inf(a + B) = a + \inf B$ and $\sup(a + B) = a + \sup B$.
(ii) $\inf aB = a \inf B$ and $\sup aB = a \sup B$ if $a > 0$.
(iii) $\inf aB = a \sup B$ and $\sup aB = a \inf B$ if $a < 0$.

Proof. Since $a+b \le a+\sup B$ for each $b \in B$ we have $\sup(aB) \le a+\sup B$. Let $\epsilon > 0$. Choose $b \in B$ such that $\sup B \le b+\epsilon$. Then $a+\sup B \le a+b+\epsilon \le \sup(a+B)+\epsilon$. Owing to the arbitrariness of ϵ we infer that $a + \sup B \le \sup(a + B)$. Thus $\sup(a+B) = a+\sup B$. In a similar fashion one proves that $\inf(a+b) = a+\inf B$. Thus (i) is proved.

Suppose $a > 0$. Since $b \le \sup B$ for $b \in B$ we have $ab \le a \sup B$ for $b \in B$ so $\sup(aB) \le a \sup B$. Let $\epsilon > 0$. Choose $b \in B$ such that $\sup B \le b(1 + \epsilon/a)$. Then $a \sup B \le ab(1 + \epsilon/a) = ab + \epsilon$. Owing to the arbitrariness of ϵ we infer that $a \sup B \le \sup(aB)$. Thus $\sup(aB) = a \sup B$. In a similar fashion one proves that $\inf(ab) = a \inf B$. Thus (ii) is proved.

We leave it to the reader to show that $\inf(-B) = -\sup B$ and that $\sup(-B) = -\inf B$. Using this and (ii) one easily proves (iii) □

An alternate method for defining the real numbers is as follows. In this construction a field structure is an immediate result but the order structure is obtained less directly than with Dededind cuts.

1.4. **Cauchy sequences of rational numbers.** We now give another construction of the real numbers.

We say a sequence c of rational numbers is **Cauchy** if for each positive integer n there is a positive integer N such that $|c_i - c_j| < 1/n$ provided $i \ge N$ and $j \ge N$. We say a sequence z of rational numbers is **null** if for each positive integer n there is a positive integer N such that $|z_i| < 1/n$ if $i \ge N$. We set

$$\mathcal{C} = \{c : C \text{ is a Cauchy sequence of rational numbers}\}$$

and we set

$$\mathcal{N} = \{z : z \text{ is a null sequence of rational numbers}\}.$$

It is then a simple to matter to verify that $\mathcal{C}$ has the structure of an integral domain with respect to pointwise addition and multiplication of rational numbers and that $\mathcal{N}$ is a maximal ideal in $\mathcal{C}$. The quotient space $\mathcal{C}/\mathcal{N}$ is therefore a field. One orders this quotient space by declaring an element A to be less than B if there is positive rational number r such that whenever s and t are representatives of A and B, respectively, there is a positive integer N such that

$$s_i + r \; < t_i \text{ whenever } i \geq N.$$

It is then a straightforward matter to verify that this completely orders the field $\mathcal{C}/\mathcal{N}$.

2. Limits of sequences in $\overline{\mathbb{R}}$.

Definition 2.1. Suppose a is a sequence in $\overline{\mathbb{R}}$ and $L \in \overline{\mathbb{R}}$. We say a_n **approaches** L **as** n **tends to infinity** and write

$$\lim_{n\to\infty} a_n = L$$

if one of the following holds.

(i) $-\infty < L < \infty$ and whenever $0 < \epsilon < \infty$ there is $N \in \mathbb{R}$ such that

$$n \geq N \;\Rightarrow |a_n - L| \leq \epsilon;$$

(ii) $L = -\infty$ and whenever $-\infty < R < 0$ there is $N \in \mathbb{N}$ such that

$$n \geq N \;\Rightarrow a_n \leq R;$$

(iii) $L = \infty$ and whenever $0 < R < \infty$ there is $N \in \mathbb{N}$ such that

$$n \geq N \;\Rightarrow a_n \geq R.$$

Theorem 2.1. Suppose a is a sequence in $\overline{\mathbb{R}}$. Then

$$\lim_{n\to\infty} a_n = \sup \mathbf{rng}\, a$$

if a is nondecreasing and

$$\lim_{n\to\infty} a_n = \inf \mathbf{rng}\, a$$

if a in nonincreasing.

Proof. Suppose a is a nonincreasing and $L = \sup \mathbf{rng}\, a$.

Suppose $-\infty < L < \infty$. Let $\epsilon > 0$. Choose a nonnegative integer N such that $L \leq a_N + \epsilon$. If $n \geq N$ we have $L - \epsilon \leq a_N \leq a_n \leq L$ which implies $|a_n - L| \leq \epsilon$. Thus $\lim_{n\to\infty} a_n = L$.

Suppose $L = \infty$. Let $R \in \mathbb{R}$. Choose a nonnnegative integer N such that $a_N \geq R$. If $n \geq N$ we have $a_n \geq a_N \geq R$. Thus $\lim_{n\to\infty} a_n = L$.

Suppose $L = -\infty$. Let $R \in \mathbb{R}$. Choose a nonnnegative integer N such that $a_N \leq R$. If $n \geq N$ we have $a_n \leq a_N \geq R$. Thus $\lim_{n\to\infty} a_n = L$.

One treats the case when a is nonincreasing in a similar fashion; we leave the details to the reader. □

As an example of the above, we have the following.

Theorem 2.2. Suppose $0 < r < 1$. Then $\lim_{n\to\infty} r^n = 0$.

Proof. **Proof One.** Suppose $\epsilon > 0$. Since $1-r > 0$ we may invoke the Archimedean Property of the real numbers to choose a positive integer M such that $1/(M+1) < 1 - r$. Using the Archimedean property once again we obtain a positive integer N such that $N \geq M/\epsilon$. We will show that

$$n \in \mathbb{N} \text{ and } n \geq N \;\Rightarrow\; r^n < \epsilon.$$

Note that $r < M/(M+1)$. For any positive integer n we use the binomial theorem to infer that

$$\left(\frac{M+1}{M}\right)^n \geq \frac{M^n + nM^{n-1}}{M^n} = 1 + \frac{n}{M}$$

so

$$r^n < \left(\frac{M}{M+1}\right)^n \leq \frac{1}{1+\frac{n}{M}} < \frac{M}{n}.$$

It follows that

$$r^n < \frac{M}{n} \leq \frac{M}{N} \leq \epsilon \quad \text{if } n \geq N.$$

□

Proof. **Proof Two. This proof will not use the Archimedean property.** Let $a_n = r^n$ for $n \in \mathbb{N}$. Then a is a decreasing sequence in $(0,1)$ so

$$L = \inf \mathbf{rng}\, a = \lim_{n\to\infty} a_n$$

exists and $L \in [0,1)$. Since

$$\mathbf{rng}\, a = \{r^n : n \in \mathbb{N}\} = \{1\} \cup \{r^{n+1} : n \in \mathbb{N}^+\} = \{1\} \cup \mathbf{rng}\, ra$$

we find that $\inf \mathbf{rng}\, ra = \inf \mathbf{rng}\, a$. Making use of previous work we find that

$$rL = r \inf \mathbf{rng}\, a = \inf(r\mathbf{rng}\, a) = \inf \mathbf{rng}\, ra = \inf \mathbf{rng}\, a = L.$$

Were it the case that $L \neq 0$ we would have $r = 1$ so $L = 0$. □

Definition 2.2. Suppose a is a sequence in $\overline{\mathbb{R}}$. We let

$$\liminf_{n\to\infty} a_n = \sup_n \inf_{m\geq n} a_m$$

and we let

$$\limsup_{n\to\infty} a_n = \inf_n \sup_{m\geq n} a_m.$$

It is very important to bear in mind that

$$\mathbb{N} \ni n \mapsto \inf\{a_m : m \geq n\} \quad \text{is nondecreasing}$$

and that

$$\mathbb{N} \ni n \mapsto \sup\{a_m : m \geq n\} \quad \text{is nonincreasing.}$$

2.1. **Relating** $\lim$**,** $\liminf$ **and** $\limsup$**.** Throughout this subsection we fix a sequence a in $\overline{\mathbb{R}}$ and let

$$\underline{l} = \liminf_{n\to\infty} a_n \quad \text{and} \quad \bar{l} = \limsup_{n\to\infty} a_n.$$

Theorem 2.3. $\underline{l} \leq \bar{l}$.

Proof. For each $n \in \mathbb{N}$ we let $A_n = \{a_m : m \geq n\}$. We have

$$\text{(1)} \qquad \inf A_n \leq \sup A_n,\ n \in \mathbb{N}.$$

I claim that

$$\text{(2)} \qquad \inf A_{n_1} \leq \sup A_{n_2}, \quad n_1, n_2 \in \mathbb{N}.$$

If $n_1 = n_2$ then (2) follows from (1). If $n_1 < n_2$ then $\inf A_{n_1} \leq \inf A_{n_2}$ because $A_{n_2} \subset A_{n_1}$ and $\inf A_{n_2} \leq \sup A_{n_1}$ by (1) so (2) holds. If $n_1 > n_2$ then $\inf A_{n_1} \leq \sup A_{n_1}$ by (1) and $\sup A_{n_1} \leq \sup A_{n_2}$ because $A_{n_1} \subset A_{n_2}$ so (2) holds.

Let $I = \{\inf A_n : n \in \mathbb{N}\}$ and let $S = \{\sup A_n : n \in \mathbb{N}\}$. Since each member of I is less than or equal each member of S we infer that $\underline{l} = \sup I \leq \inf S = \bar{l}$. □

Proposition 2.1. Suppose $-\infty < l < \infty$. Then following statements hold:

(i) $l \leq \underline{l}$ if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $l < a_n + \epsilon$ whenever $n \geq N$.

(ii) $\underline{l} \leq l$ if and only if for each $\epsilon > 0$ and each $N \in \mathbb{N}$ there is $n \geq N$ such that $a_n \leq l + \epsilon$.

(iii) $l \leq \bar{l}$ if and only if for each $\epsilon > 0$ and each $N \in \mathbb{N}$ there is $n \geq N$ such that $l \leq a_n + \epsilon$.

(iv) $\bar{l} \leq l$ if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $a_n < l + \epsilon$ whenever $n \geq N$.

Proof. Exercise for the reader. □

Proposition 2.2. The following statements hold.

(i) $\underline{l} = -\infty$ if and only if for all $R \in \mathbb{R}$ and all $N \in \mathbb{N}$ there exist $n \in \mathbb{N}$ such that $n \geq N$ and $a_n \leq R$.

(ii) $\underline{l} = \infty$ if and only if for all $R \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $R \leq a_n$ if $n \geq N$.

(iii) $\bar{l} = \infty$ if and only if for all $R \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n \geq R$ for some $n \geq N$.

(iv) $\bar{l} = -\infty$ if and only if for all $R \in \mathbb{R}$ there is $N \in \mathbb{N}$ $a_n \leq R$ if $n \geq N$.

Proof. Exercise for the reader. □

Theorem 2.4. Suppose $L \in \overline{\mathbb{R}}$. Then

$$\underline{l} = L = \bar{l} \;\Leftrightarrow\; \lim_{n\to\infty} a_n = L.$$

Proof. Exercise for the reader. □

Definition 2.3. We say a sequence a in $\mathbb{R}$ is **Cauchy** if for each $\epsilon > 0$ there is a nonnegative integer N such that

$$m, n \in \mathbb{N} \text{ and } m, n \geq N \;\Rightarrow |a_m - a_n| \leq \epsilon.$$

Theorem 2.5. Suppose a is a sequence in $\mathbb{R}$. The following are equivalent:

(i) $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$.

(ii) a is Cauchy.

Remark 2.1. We will give a proof that is independent of the material on lim inf and lim sup. Note that L in the proof is none other than $\liminf_{n\to\infty} a_n$.

Proof. Suppose (i) holds. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon/2$ if $n \geq N$. Then if $m, n \geq N$ we have

$$|a_m - a_n| \leq |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus a is Cauchy.

Now suppose (ii) holds. For each $n \in \mathbb{N}$ let $T_n = \{a_m : m \in \mathbb{N} \text{ and } m \geq n\}$ and let $I_n = \inf T_n$. Since $T_m \subset T_n$ if $m, n \in \mathbb{N}$ and $m \geq n$ we find that I is nondecreasing. Let $L = \sup\{I_n : n \in \mathbb{N}\}$.

Suppose $\epsilon > 0$ and suppose $0 < \eta < \epsilon$. Choose $N \in \mathbb{N}$ such that $m, n \in \mathbb{N}$,

$$m \geq N \text{ and } n \geq N \;\Rightarrow\; |a_m - a_n| < \eta.$$

Suppose $n \in \mathbb{N}$ and $n \geq N$. Then

$$a_m \geq a_n - \eta \quad \text{if } m \in \mathbb{N} \text{ and } m \geq N.$$

This implies

$$L \geq I_n \geq a_n - \eta.$$

We also have

$$I_m \leq a_m \leq a_n + \eta \quad \text{if } m \in \mathbb{N} \text{ and } m \geq N.$$

Since I is nondecreasing, this implies

$$L \leq a_n + \eta.$$

We infer that $-\infty < L < \infty$ and that $|L - a_n| \leq \eta < \epsilon$, as desired. □

Exercise 2.1. Suppose a and b are sequences in $\overline{R}$. Show that

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty} a_n + b_n$$

and that

$$\limsup_{n\to\infty} a_n + b_n \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Exhibit sequences a and b in $\overline{R}$ such that equality does not hold in either case.

Exercise 2.2. Suppose

$$a_1, a_2, \ldots, a_n, \ldots, \quad n = 1, 2, 3, \ldots$$

is a bounded sequence in $\mathbb{R}$ with limit 0 and let s be the sequence of **arithmetic means**:

$$s_n = \frac{1}{n}(a_1 + \cdots a_n), \quad n = 1, 2, 3, \ldots.$$

Show that s has limit 0.

Exercise 2.3. Let

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}, \quad n = 1, 2, 3, \ldots.$$

Show that a converges. Bonus: To what?

Exercise 2.4. Suppose a is a sequence in $\overline{R}$ with limit $L \neq 0$. Show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}.$$

3. The uncountability of the real numbers.

Theorem 3.1. Suppose $r \in \mathbb{R} \sim \{1\}$. Then

$$\sum_{m=0}^{n} r^m = \frac{1 - r^{n+1}}{1 - r}.$$

for any nonnegative integer n.

Proof. The Theorem holds trivially if $n = 0$. Let us suppose, inductively, that the n is a nonnegative integer for which the Theorem holds. Then

$$\sum_{m=0}^{n+1} r^m = \sum_{m=0}^{n} r^m + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{(n+1)+1}}{1 - r}.$$

□

Corollary 3.1. If s is a sequence in $\{0, 1\}$ then

$$\sup \left\{ \sum_{n=0}^{N} \frac{s_n}{2^{n+1}} : N \in \mathbb{N} \right\} \leq 1.$$

Definition 3.1. A basic map. We let

$$\mathcal{S}$$

be the set of mappings $s : \mathbb{N} \to \{0, 1\}$ such that $\{n \in \mathbb{N} : s(n) = 1\}$ is infinite and we define

$$\Phi : \mathcal{S} \to \overline{\mathbb{R}}$$

by letting

$$\Phi(s) = \sup \left\{ \sum_{n=0}^{N} \frac{s_n}{2^{n+1}} : N \in \mathbb{N} \right\} \quad \text{for } s \in \mathcal{S}.$$

Remark 3.1. So if for a sequence a of real numbers we *define*

$$\sum_{n=0}^{\infty} = \lim_{N \to \infty} \sum_{n=0}^{N} a_n$$

we have

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{s_n}{2^n}.$$

Proposition 3.1. We have

$$0 < \Phi(s) \leq 1 \quad \text{for and } s \in \mathcal{S}.$$

Proof. This follows immediately from the preceding Theorem and the fact that for any $s \in \mathcal{S}$ we have $s(n) = 1$ for some $n \in \mathbb{N}$. □

We endow $\mathcal{S}$ with the **dictionary order** by declaring $s < t$ for $s, t \in \mathcal{S}$ if $s \neq t$ and $s_n = 0$ and $t_n = 1$ where n is the least member of $\{p \in \mathbb{N} : s_p \neq t_p\}$.

Proposition 3.2. $<$ is a linear ordering of $\mathcal{S}$.

Proof. It follows immediately from the definition that $<$ is trichotomous.

Suppose $s, t, u \in \mathcal{S}$, $s < t$ and $t < u$. Let n be the least member of $\{p \in \mathcal{N} : s_p \neq t_p\}$ and let o be the least member of $\{p \in \mathcal{N} : t_p \neq u_p\}$. Then $s_n = 0$ and $t_n = 1$ and $t_o = 0$ and $u_o = 1$. In particular, it is impossible that $n = o$. In case $n < o$ we have $s_p = t_p = u_p$ for $p < n$, $s_n = 0$ and $t_n = 1$, and $t_n = u_n$ so $s < u$. In case $o < n$ we have $s_p = t_p = u_p$ for $p < o$, $s_o = t_o$ and $t_o = 0$ and $u_o = 1$ so $s < u$. Thus $<$ is transitive. □

Proposition 3.3. Suppose $s, t \in \mathcal{S}$. Then $s < t$ if and only if $\Phi(s) < \Phi(t)$.

Proof. Let o be the least member of $\{p \in \mathbb{N} : s_p \neq t_p\}$ and let

$$B = \sum_{n<o} \frac{s_n}{2^{n+1}}.$$

Then

$$\Phi(s) = B + \frac{s_o}{2^{o+1}} + S \quad \text{and} \quad \Phi(t) = B + \frac{t_o}{2^{o+1}} + T$$

where we have set

$$S = \sup \left\{ \sum_{n=o+1}^{N} \frac{s_n}{2^{n+1}} : N \in \mathbb{N} \text{ and } o + 1 \leq N \right\}$$

and

$$T = \sup \left\{ \sum_{n=o+1}^{N} \frac{t_n}{2^{n+1}} : N \in \mathbb{N} \text{ and } o + 1 \leq N \right\}.$$

Then $S \leq 1/2^{o+1}$ by Theorem 3.1 and $T > 0$ since $\{n \in \mathbb{N} : n > o + 1 \text{ and } t_n \neq 0\}$ is nonempty.

Suppose $s < t$. Then $s_o = 0$ and $t_o = 1$ so

$$\Phi(s) = B + S < B + \frac{1}{2^{o+1}} + T = \Phi(t).$$

Suppose $\Phi(s) < \Phi(t)$. Then $s \neq t$. By the trichotomy of $<$ we have either $s < t$ or $t < s$. Were it the case that $t < s$ we would have $\Phi(t) < \Phi(s)$ by what we proved in the previous paragraph. So $s < t$ □

Proposition 3.4. $\mathbf{rng}\,\Phi = (0, 1]$.

Proof. We have already shown that $\mathbf{rng}\,\Phi \subset (0, 1]$.

Suppose $a \in (0, 1]$. We define the sequence $p : \mathbb{N} \to \mathbb{N}$ by induction as follows. p_0 is the least member of $\mathbb{N}$ such that $2^{-m-1} < a$. If $n \in \mathbb{N}$ and p_j, $j \in \mathbb{I}(n)$, are defined we let p_n be the least member m of $\mathbb{N}$ such that

$$\sum_{j \in \mathbb{I}(n)} 2^{-(p_j+1)} + 2^{-(m+1)} < a.$$

We let $s \in \mathcal{S}$ be such that $\{n \in \mathbb{N} : s_n = 1\} = \mathbf{rng}\,p$. We leave it to the reader to us the Archimedean property of the real numbers to verify that $\Phi(s) = a$. □

We are now able to prove the following basic result.

Theorem 3.2. $\mathbb{R} \approx 2^{\mathbb{N}}$.

Proof. Since

$$\mathbb{Z} \times \mathcal{S} \ni (z, s) \mapsto z + \Phi(s) \in \mathbb{R}$$

is univalent with range $\mathbb{R}$ we have

$$\mathbb{Z} \times \mathcal{S} \approx \mathbb{R}. \tag{1}$$

Since $\{A \in 2^{\mathbb{N}} : A \text{ is finite}\}$ is countable we have $\mathcal{S} \approx 2^{\mathbb{N}}$ by a previous Theorem. Moreover,

$$\{(0, 0)\} \cup \{(2m, m) : m \in \mathbb{N}^{+}\} \cup \{(2m + 1, -m) : m \in \mathbb{N}^{+}\}$$

is univalent with domain $\mathbb{N}$ and range $\mathbb{Z}$ so $\mathbb{N} \approx \mathbb{Z}$. Thus

$$\mathbb{Z} \times \mathcal{S} \approx \mathbb{N} \times 2^{\mathbb{N}}. \tag{2}$$

Next we show that

$$\mathbb{N} \times 2^{\mathbb{N}} \approx 2^{\mathbb{N} \times \mathbb{N}}. \tag{3}$$

For $(n, A) \in \mathbb{N} \times 2^{\mathbb{N}}$ we let $f(n, A) = \{n\} \times A \in 2^{\mathbb{N} \times \mathbb{N}}$. For $C \subset \mathbb{N} \times \mathbb{N}$ we let $g(C) = (0, \beta[C]) \in \mathbb{N} \times 2^{\mathbb{N}}$ where β is a univalent function on $\mathbb{N} \times \mathbb{N}$ with range equal $\mathbb{N}$. Since f and g are univalent, (3) follows from the Schroeder-Bernstein Theorem. Finally,

$$2^{\mathbb{N} \times \mathbb{N}} \ni C \mapsto \beta[C] \in 2^{\mathbb{N}}$$

is univalent with range $2^{\mathbb{N}}$ so

$$2^{\mathbb{N} \times \mathbb{N}} \approx 2^{\mathbb{N}}. \tag{4}$$

Combining (1),(2),(3) and (4) we obtain the desired result. □