**Definition.** Suppose $V_1, \ldots, V_m$ and $W$ are vector spaces. We say a function

$$\mu : V_1 \times \cdots \times V_m \to W$$

is **multilinear** if it is linear in each of its $m$ arguments when the other $m-1$ are held fixed. Let

$$\mathbf{L}(V_1, \ldots, V_m; W)$$

be the set of such $\mu$. Note that $\mathbf{L}(V_1, \ldots, V_m; W)$ is a linear subspace of the vector space of all $W$-valued functions on $V_1 \times \cdots \times V_m$ and is thus a vector space with respect to pointwise addition and scalar multiplication.

Suppose $\omega_i \in V_i^*,\ i = 1, \ldots, m$ and $w \in W$. Define

$$\omega_1 \ldots \omega_m w : V_1 \times \cdots \times V_m \to W$$

to have the value $\omega_1(v_1) \cdots \omega_m(v_m) w$ at $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ and note that

$$\omega_1 \ldots \omega_m w \in \mathbf{L}(V_1, \ldots, V_m; W).$$

In case $W = \mathbf{R}$ and $w = 1$ one customarily writes

$$\omega_1 \cdots \omega_m$$

for $\omega_1 \cdots \omega_m w$.

**Problem 1.** Suppose for each $i = 1, 2$, $V_i$ is a finite dimensional vector space of dimension $n_i$ and with ordered basis $v_i$. Let $\mu \in \mathbf{L}(V_1, V_2; \mathbf{R})$ and let $A \in \mathbf{M}_{n_2}^{n_1}$ be such that

$$A(i, j) = \mu(v_i, v_j),\ i = 1, \ldots, n_1,\ j = 1, \ldots, n_2.$$

Show that there are $\omega_i \in V_i^*,\ i = 1, 2$, such that $\mu = \omega_1 \omega_2$ if and only if the rank of $A$ does not exceed 1.

**Problem 2.** Suppose $V_1, \ldots, V_m$ and $W$ are finite dimensional. Let $B_i$ be a basis for $V_i,\ i = 1, \ldots, m$ and let $C$ be a basis for $W$. Show that

$$\mu = \sum_{(v_1, \ldots, v_m, w) \in B_1 \times \cdots \times B_m \times C\}} w^*(\mu(v_1, \ldots, v_m))\, v_1^* \cdots v_m^* w$$

for each $\mu \in \mathbf{L}(V_1, \ldots, V_m; W)$. Use this to show that

$$\{v_1^* \ldots v_m^* w : (v_1, \ldots, v_m, w) \in B_1 \times \cdots \times B_m \times C\}$$

is a basis for $\mathbf{L}(V_1, \ldots, V_m; W)$, concluding thereby that its dimension is $n_1 \cdots n_m \cdot l$.

**Definition.** Suppose now that $V_i$ has norm $|\cdot|_{V_i},\ i = 1, \ldots, m$ and that $W$ has norm $|\cdot|_W$. For each $\mu \in \mathbf{L}(V_1, \ldots, V_m; W)$ we let

$$||\mu||_{V_1, \ldots, V_m; W} = \sup\{|\mu(v_1, \ldots, v_m)|_W : v_i \in V_i \text{ and } |v_i|_{V_i} \le 1\}.$$

Very often one omits the subscripts on the norms relying on the context to resolve the resulting ambiguities.

**Problem 3.**

(1) Suppose $\mu \in \mathbf{L}(V_1, \ldots, V_m; W)$ and $M \in [0, \infty)$. Then $|\mu(v_1, \ldots, v_m)| \le M|v_1| \cdots |v_m|$ whenever $v_i \in V_i,\ i = 1, \ldots, m$ if and only if $||\mu|| \le M$.

(2) $||\mu + \nu|| \le ||\mu|| + ||\nu||$ whenever $\mu, \nu \in \mathbf{L}(V_1, \ldots, V_m; W)$;
(3) $||c\mu|| = |c|||\mu||$ whenever $c \in \mathbf{R}$ and $\mu \in \mathbf{L}(V_1, \ldots, V_m; W)$;
(4) If $\mu \in \mathbf{L}(V_1, \ldots, V_m; W)$ then $\mu$ is continuous if and only if $||\mu|| < \infty$.

**Problem 4.** Suppose $U, V, W$ are normed vector spaces, $L \in \mathbf{L}(U; V)$ and $M \in \mathbf{L}(V; W)$. Then

$$||M \circ L|| \le ||M||||L||.$$

**Problem 5.** Suppose $V$ is a finite dimensional Euclidean space. Show that the mapping

$$V \ni v \mapsto (V \ni \tilde{v} \mapsto \tilde{v} \bullet v \in \mathbf{R}) \in V^*$$

carries $V$ isomorphically onto $V^*$. This map is called the **polarity** of the inner product and we induce an inner product on $V^*$ by requiring that it be an isometry. Conversely, if $\beta$ carries $V$ isomorphically onto $V^*$ and satisfies the conditions

(i) $\beta(v)(w) = \beta(w)(v),\ v, w \in V$ and
(ii) $\beta(v)(v) > 0$ if $v \in V \sim \{\mathbf{0}\}$
then we may obtain an inner product on $V$ by setting $v \bullet w = \beta(v)(w), /v, w \in V$.

**Problem 6.** Verify that the adjoint mapping defined earlier is a linear isomorphism if $V$ and $W$ above are finite dimensional. Do this by showing that the adjoint mapping is linear (this is trivial) and that if $B$ is a basis for $V$ and $C$ is a basis for $W$ then $\{v^*w : v \in B \text{ and } w \in C\}$ is a basis for $\mathbf{L}(V, W)$; $\{wv^*; v \in B \text{ and } w \in C\}$ is a basis for $\mathbf{L}(W^*, V^*)$; and

$$(v^*w)^* = wv^* \quad \text{whenever } v \in B \text{ and } w \in C.$$

(Here we have written $w$ instead of $\iota(w)$ for $w \in W$ as we indicated we might do so when $\iota$ was defined.)

**Definition.** Let $V$ and $W$ be finite dimensional Euclidean spaces with polarities $\beta$ and $\gamma$, respectively. Let

$$^* = \beta^{-1} \circ (\cdot^*) \circ \gamma$$

where the $^*$ on the right is the adjoint introduced previously and where the one on the left is being introduced now. Note that $^*$ (on the left!), also called the **adjoint** (sorry about that, you were warned!) carries $\mathbf{L}(V; W)$ isomorphically onto $\mathbf{L}(W; V)$. Verify that if $L \in \mathbf{L}(V; W)$ and $K \in \mathbf{L}(W; V)$ then

$$L(v) \bullet w = v \bullet K(w) \quad \text{whenever } v \in V,\ w \in W \quad \Leftrightarrow K = L^*.$$

Verify that, under appropriate hypotheses,

$$(L \circ M)^* = M^* \circ L^*.$$

Note an additional and rather significant ambiguity in the notation. If $L : V \to W$ and $W$ is a subspace of the inner product space $Z$ then we have $L^* \in \mathbf{L}(Z; V)$ (same $L$ but *two* $L^*$'s!). This same ambiguity was present when we first encountered the adjoint.

**Problem 7.** Suppose $V$ and $W$ are finite dimensional Euclidean spaces and $L \in \mathbf{L}(V; W)$. Then $||L||$ is the square root of the largest eigenvalue of $L^* \circ L$.

**Problem 8.** Suppose $V$ is a finite dimensional vector space. Let

$$\zeta : \mathbf{L}(V; V) \to \mathbf{L}(V^*, V; \mathbf{R})$$

be such that

$$\zeta(L)(\omega, v) = \omega(L(v)), \quad \omega \in V^*,\ v \in V.$$

Verify that $\zeta$ is linear. Verify that it is an isomorphism by showing that if $B$ is a basis for $V$ then $\zeta$ carries the basis $\tilde{v}^* v$ of $\mathbf{L}(V;V)$ to the basis $\iota(\tilde{v})v^*$ of $\mathbf{L}(V^*, V; \mathbf{R})$.

**Definition.** Suppose $V_1, \ldots, V_m$ are finite dimensional vector spaces. We set

$$V_1 \otimes \cdots \otimes V_m = \mathbf{L}(V_1^*, \ldots, V_m^*; \mathbf{R})$$

and call this vector space the **tensor product** of $V_1, \ldots, V_m$. For each $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ we set

$$v_1 \otimes \cdots \otimes v_m = \iota(v_1) \cdots \iota(v_m) \in V_1 \otimes \cdots \otimes V_m$$

and note that

$$V_1 \times \cdots \times V_m \ni (v_1, \ldots, v_m) \mapsto v_1 \otimes \cdots \otimes v_m \in V_1 \otimes \cdots \otimes V_m$$

is multilinear.

**Problem 9.** Show that if $V_1, \ldots, V_m$ are finite dimensional vector spaces and $W$ is a vector space then

$$\mathbf{L}(V_1 \otimes \cdots \otimes V_m; W) \ni \mu \mapsto (V_1 \times \cdots \times V_m \ni (v_1, \ldots, v_m) \mapsto \mu(v_1 \otimes \cdots \otimes v_m) \in W) \in \mathbf{L}(V_1, \ldots, V_m; W)$$

carries $\mathbf{L}(V_1 \otimes \cdots \otimes V_m; W)$ isomorphically onto $\mathbf{L}(V_1, \ldots, V_m; W)$. In particular, for any $\tilde{\mu} \in \mathbf{L}(V_1, \ldots, V_m; W)$ there is one and only $\mu \in \mathbf{L}(V_1 \otimes \cdots \otimes V_m; W)$ such that

$$\tilde{\mu}(v_1, \ldots, v_m) = \mu(v_1 \otimes \cdots \otimes v_m), \quad v_i \in V_i, \ i = 1, \ldots, m.$$

This is called the **universal property of the tensor product.**

**Problem 10.** Suppose $V$ and $W$ are finite dimensional vector spaces. By the universal property of the tensor product there is a unique linear map

$$\gamma : V^* \otimes W \to \mathbf{L}(V; W)$$

such that $\gamma(\omega \otimes v) = \omega v$ whenever $\omega \in V^*$ and $w \in W$. Show that $\gamma$ is an isomorphism by finding a basis of $V^* \otimes W$ which is carried to a basis of $\mathbf{L}(V; W)$ by $\gamma$.

**Problem 11.** Suppose $V$ and $W$ are finite dimensional Euclidean spaces. Verify that

$$\mathbf{L}(V;W) \times \mathbf{L}(V;W) \ni (K, L) \mapsto \mathbf{trace}\,(K^* \circ L)$$

is an inner product.

Verify that

$$|L| \leq \sqrt{\mathbf{dim}\, V}\,||L|| \quad \text{and that} \quad ||L|| \leq |L|.$$

Note that equality occurs in the left hand inequality if $L^* = L^{-1}$ which is to say if $L$ is orthogonal. Note that equality occurs in the right hand inequality if $\mathbf{dim}\, V = 1$.

**Problem 12.** Suppose $V$ and $W$ are finite dimensional Euclidean spaces. Suppose $L \in \mathbf{L}(V;W)$. Show that

$$||L^*|| = ||L||.$$

Do this by first showing that

$$||L|| = \sup\{|L(v) \bullet w| : v \in V, \ |v| \leq 1, \ w \in W, \ |w| \leq 1\}.$$