Submanifolds.

Let n be a positive integer.

Definition. We say f is an n**-diffeomorphism** if
(1) f is function whose domain and range of f are open subsets of $\mathbf{R}^n$;
(2) f is smooth;
(3) f is univalent and, for each $x \in \mathbf{dmn}\, f$, $\partial f(x)$ carries $\mathbf{R}^n$ isomorphically onto itself.
Whenever U and V are open subsets of $\mathbf{R}^n$ we let

$$\mathbf{Diffeo}_n$$

be the set of ordered triples (U, F, V) such that F is an n-diffeomorphism with domain U and range V.

Proposition. We have
(1) $\emptyset$ is an n-diffeormorphism;
(2) if F is an n-diffeormorphism and W is an open subset of $\mathbf{R}^n$ then $F|W$ is an n-diffeormorphism;
(3) if $\mathcal{U}$ is a family of open subsets of $\mathbf{R}^n$, $F : \bigcup \mathcal{U} \to \mathbf{R}^n$, F is univalent and $F|U$ is a n-diffeormorphism for each $U \in \mathcal{U}$ then F is an n-diffeormorphism;
(4) if F is an n-diffeormorphism then F^{-1} is an n-diffeomorphism;
(5) if F, G are n-diffeomorphisms then $F \circ G$ is an n-diffeormorphism.

Proof. Exercise for the reader. It will be necessary to use the Inverse Function Theorem and its Corollaries, the Chain Rule and the fact the inversion on $\mathbf{GL}\,(\mathbf{R}^n)$ is smooth. □

Suppose m is an integer and $0 \le m \le n$.

Definition.

$$\mathbf{R}^{m,n} = \{x \in \mathbf{R}^n : x_i = 0 \text{ whenever } i < m \le n\}.$$

Let

$$\mathbf{p}_{m,n},\ \mathbf{q}_{m,n},\ \mathbf{i}_{m,n},\ \mathbf{j}_{m,n}$$

be defined by the following requirements:

$$\begin{aligned} \mathbf{p}_{m,n} &: \mathbf{R}^n \to \mathbf{R}^m, \\ \mathbf{q}_{m,n} &: \mathbf{R}^n \to \mathbf{R}^{n-m}, \\ \mathbf{i}_{m,n} &: \mathbf{R}^m \to \mathbf{R}^{m,n}, \\ \mathbf{j}_{m,n} &: \mathbf{R}^{n-m} \to \left(\mathbf{R}^{m,n}\right)^{\perp}; \end{aligned}$$

if $m = 0$ then

$$\mathbf{p}_{m,n} = 0 \quad \text{and} \quad \mathbf{q}_{m,n} = \mathbf{i}_{\mathbf{R}^n};$$

if $1 < m < n$ then

$$\mathbf{p}_{m,n}(x) = \sum_{i=1}^{m} x_i \mathbf{e}_i,\ x \in \mathbf{R}^m \quad \text{and} \quad \mathbf{q}_{m,n}(y) = \sum_{j=1}^{n-m} y_j \mathbf{e}_{m+j},\ y \in \mathbf{R}^{n-m};$$

if $m = n$ then

$$\mathbf{p}_{m,n} = \mathbf{i}_{\mathbf{R}^n} \quad \text{and} \quad \mathbf{q}_{m,n} = 0;$$

and

$$\mathbf{i}_{\mathbf{R}^n} = \mathbf{i}_{m,n} \circ \mathbf{p}_{m,n} + \mathbf{j}_{m,n} \circ \mathbf{q}_{m,n}.$$

We let

$$\mathbf{U}^n = \{x \in \mathbf{R}^n : |x| < 1\}$$

and we let

$$\mathbf{U}^{m,n} = \mathbf{U}^n \cap \mathbf{R}^{m,n}.$$

Whenever $m \ge 1$ we let

$$\mathbf{U}^{m,n,+} = \{x \in \mathbf{U}^{m,n} : x_m > 0\}.$$

Definition. Suppose V is an open subset of $\mathbf{R}^n$. We let

$$\mathbf{M}_m(V)$$

be the family of nonempty subsets M of V such that

(1) if $a \in M$ there is $(\mathbf{U}^n, \Phi, U) \in \mathbf{Diffeo}_n$ such that $a \in U \subset V$, $\Phi(0) = a$ and $U \cap M = \Phi[\mathbf{U}^{m,n}]$.

(2) if $m \ge 1$ and $b \in (V \sim \mathbf{cl}\, M) \sim M$ there is $(\mathbf{U}^n, \Phi, U) \in \mathbf{Diffeo}_n$ such that $b \in U \subset V$, $\Phi(0) = b$ and $U \cap M = \Phi[\mathbf{U}^{m,n,+}]$.

We call the members of $\mathbf{M}_m(V)$ **smooth m-dimensional submanifolds of V.**

For each $M \in \mathbf{M}_{m,n}(V)$ we set

$$\partial M = (V \cap \mathbf{cl}\, M) \sim M.$$

Theorem. Suppose V is an open subset of $\mathbf{R}^n$ and M is a nonempty subset of V. Then

(1) $M \in \mathbf{M}_0(V)$ if and only if M is a nonempty subset of V which meets any compact subset of V in a finite set.

(2) if $M \in \mathbf{M}_0(V)$ then $\partial M = \emptyset$.

(3) $M \in \mathbf{M}_n(V)$ and $\partial M = \emptyset$ if and only if each connected component of M is a connected component of V.

(4) if $m \ge 1$ and $M \in \mathbf{M}_m(M)$ then $\partial M \in \mathbf{M}_{m-1}(V)$ and $\partial(\partial M) = \emptyset$.

Proof. These are straightforward consequences of the definitions. □

Theorem. Suppose $1 \le m < n$, V is an open subset of $\mathbf{R}^n$ and M is a nonempty subset of V. Then $M \in \mathbf{M}_m(V)$ if and only if

(1) for each $a \in M$ there are an open subset U of V and a smooth map $F : U \to \mathbf{R}^{n-m}$ such that

$$\mathbf{dim}\,\mathbf{rng}\,\partial f(a) = n - m$$

and

$$M \cap U = \{x \in V : F(x) = F(a)\};$$

(2) for each $b \in (V \cap \mathbf{cl}\, M) \sim M$ there are an open subset U of V and smooth maps

$$F : U \to \mathbf{R}^{n-m} \quad \text{and} \quad g : U \to \mathbf{R}$$

such that

$$\mathbf{dim}\,\mathbf{rng}\,\partial f(a) = n - m, \qquad \partial g(a) \notin \mathbf{span}\,\{\partial F^i(a) : i = 1, \ldots, n-m\}$$

and

$$U \cap M = \{x \in U : F(x) = F(a) \text{ and } g(x) > g(a)\}.$$

Remark. Note that if (2) holds there is an open subset T of U such that $a \in T$ and

$$T \cap \mathbf{cl}\, M = \{x \in T : F(x) = F(a) \text{ and } g(x) \ge g(a)\}$$

and

$$T \cap \partial M = \{x \in T : F(x) = F(a) \text{ and } g(x) = g(a)\}.$$

Proof. Exercise for the reader. Use the Implicit Function Theorem. □

Theorem. Suppose V is an open subset of $\mathbf{R}^n$ and M is a nonempty subset of V. Then $M \in \mathbf{M}_n(V)$ if and only if for each $b \in (V \cap \mathbf{cl}\, M) \sim M$ there are an open subset U of V and a smooth map

$$g : U \to \mathbf{R}$$

such that

$$\partial g(a) \neq 0$$

and

$$U \cap \mathbf{cl}\, M = \{x \in U : g(x) > g(a)\}.$$

Remark. Note that if g is as above then there is an open subset T of U such that $a \in T$ and

$$T \cap \partial M = \{x \in T : g(x) = g(a)\}.$$

Proof. Exercise for the reader. Use the Implicit Function Theorem. $\square$

Immersions.

Definition. Suppose T is an open subset of $\mathbf{R}^m$ and V is an open subset of $\mathbf{R}^n$. By a **proper immersion of T into V** we mean a smooth univalent map $\phi : T \to V$ such that

$$\mathbf{dim}\,\mathbf{rng}\,\partial\phi(t) = m \quad \text{whenever } t \in T$$

and

$$\phi^{-1}[K] \text{ is a compact subset of } T \text{ whenever } K \text{ is a compact subset of } V.$$

We let

$$\mathbf{Imm}_{m,n}$$

be the set of ordered triples (T, ϕ, V) such that T is an open subset of $\mathbf{R}^m$, V is an open subset of $\mathbf{R}^n$ and ϕ is a proper immersions of T into V.

Theorem. Suppose T is an open subset of $\mathbf{R}^m$, V is an open subset of $\mathbf{R}^n$ and $\phi : T \to V$ is a smooth univalent map such that

$$\mathbf{dim}\,\mathbf{rng}\,\partial\phi(t) = m \quad \text{whenever } t \in T.$$

Let $M = \mathbf{rng}\,\phi$. Then the following conditions are equivalent:

(1) $$(T, \phi, U) \in \mathbf{Imm}_m.$$

(2) $$M \in \mathbf{M}_m(V) \text{ and } \partial M = \emptyset.$$

Proof. Let $M = \mathbf{rng}\,\phi$.

Part One. Suppose (1) holds and $a \in V \cap \mathbf{cl}\, M$. Let

$$\mathcal{K} = \{\phi^{-1}[\mathbf{B}_a(r)] : 0 < r < \infty \text{ and } \mathbf{B}_a(r) \subset V\}$$

Then $\mathcal{K}$ is a nested family of nonempty compact subsets of T any point of whose nonvoid intersection is carried to a by ϕ. Since ϕ is univalent there is $c \in T$ such that $\bigcap \mathcal{K} = \{c\}$ and $\phi(c) = a$. Thus

$$\text{for any open subset } S \text{ of } T \text{ such that } c \in S \text{ there is } r > 0 \text{ such that } \phi^{-1}[\mathbf{U}_a(r)] \subset S.$$

In particular,

$$V \cap \mathbf{cl}\, M = V \cap M.$$

Choose $l \in \bigotimes(\mathbf{R}^{n-m}, \mathbf{R}^n)$ such that

$$\mathbf{rng}\,\partial\phi(c) + \mathbf{rng}\,l = \mathbf{R}^n.$$

and let

$$G(t,u) = \phi(t) + l(u),\ (t,u) \in T \times \mathbf{R}^{n-m}.$$

Since $\mathbf{rng}\,\partial G(c,0) = \mathbf{R}^n$ we may apply the Inverse Function Theorem to obtain open subsets S of T and W of $\mathbf{R}^{n-m}$ such that $(c,0) \in S \times W$ and $H = G|(S \times W)$ is an n-diffeomorphism. Now choose $r > 0$ such that if $U = \mathbf{U}_a(r)$ then

$$U \subset \mathbf{rng}\,H \text{ and } \phi^{-1}[U] \subset S.$$

Let $q(t,u) = u$ for $(t,u) \in \mathbf{R}^m \times \mathbf{R}^{n-m}$ and set

$$F = \big(q \circ H^{-1}\big)|U.$$

Since $F(H(t,u)) = u$ for whenever $(t,u) \in S \times W$ and $H(t,u) \in U$ we find that

$$\mathbf{rng}\,\partial F(c) = n - m.$$

Suppose $x \in M \cap U$. Then $x = \phi(t)$ for some $t \in S$. Since $H(t,0) = x$ we find that $F(x) = 0$. Thus

$$\{x \in U : F(x) = F(a)\} = M \cap U$$

and (2) holds.

Part Two. Suppose (2) holds and $a \in M$. It will suffice to show that there is $r > 0$ such that $\phi^{-1}[\mathbf{B}_a(r)]$ is a compact subset of T. Let $(\mathbf{U}^n, \Phi, U) \in \mathbf{Diffeo}_n$ be such that $a \in U \subset V$

$$U \cap M = \Phi[\mathbf{U}^{m,n}].$$

Set $S = \phi^{-1}[U]$ and set $\psi = (\mathbf{p}_{m,n} \circ \Phi^{-1} \circ \phi)|S$. Note that

$$\mathbf{rng}\,\partial\psi(t) = \mathbf{R}^m,\ t \in S.$$

Thus $(S, \psi, \psi[S])$ is an m-diffeomorphism by earlier results. It follow that $\psi^{-1}[L]$ is a compact subset of S whenever L is a compact subset of $\psi[S]$. Choose $r > 0$ such that $\mathbf{B}_a(r) \subset U$. Then $L = \mathbf{p}_{m,n} \circ \Phi^{-1}[\mathbf{B}_a(r)]$ is a compact subset of $\psi[S]$ and

$$\phi^{-1}[\mathbf{B}_a(r)] = \psi^{-1}[L]$$

so $\phi^{-1}[\mathbf{B}_a(r)]$ is a compact subset of S, as desired. □

Theorem. Suppose

$$(T_i, \phi_i, V_i) \in \mathbf{Imm}_m,\ i = 1,2$$

and

$$V_2 \cap \mathbf{rng}\,\phi_1 = V_1 \cap \mathbf{rng}\,\phi_2.$$

Then

$$(\phi_1^{-1}[V_2], \phi_2^{-1} \circ \phi_1, \phi_2^{-1}[V_1]) \in \mathbf{Diffeo}_m.$$

Proof. Suppose $c_1 \in \phi^{-1}[V_2]$. Since $\mathbf{rng}\,\phi_1 \in \mathbf{M}_m(V_1)$ we may choose $(\mathbf{U}^n, \Phi, U) \in \mathbf{Diffeo}_n$ such that $\phi_1(t_1) \in U \subset V_1 \cap V_2$ and such that $U \cap \mathbf{rng}\,\phi_1 = \Phi[\mathbf{U}^{m,n}]$. Let

$$\psi_i = \mathbf{p}_{m,n} \circ \Phi^{-1} \circ \phi_i, i = 1,2.$$

Evidently, ψ_i is a smooth univalent map carrying the open subset $\phi_i^{-1}[U_i]$ of $\mathbf{R}^m$ onto the open subset $(\mathbf{p}_{m,n} \circ \Phi^{-1})[U]$ of $\mathbf{R}^m$, $i = 1,2$ and

$$(\phi_2^{-1} \circ \phi_1)|\phi_1^{-1}[U] = \psi_2^{-1} \circ \psi_1.$$

Since $c_1 \in \phi_1^{-1}[U]$ the proof will be complete if we can show that

$$(\phi_i^{-1}[U], \psi_i, (\mathbf{p}_{m,n} \circ \Phi^{-1})[U]) \in \mathbf{Diffeo}_m,\ i = 1, 2,$$

and this will follow if we can show that

$$\mathbf{rng}\, \partial\psi_i(t_i) = \mathbf{R}^m,\ t_i \in \phi_i^{-1}[U],\ i = 1, 2.$$

So suppose $i \in \{1, 2\}$ and $t_i \in \phi_i^{-1}[U]$. Then $\mathbf{dim}\,\mathbf{rng}\, \partial(\Phi^{-1} \circ \phi_i)(t) = m$ by the Chain Rule. But as the range of $\Phi^{-1} \circ \phi_i$ is a subset of $\mathbf{R}^{m,n}$ we find that

$$\mathbf{rng}\, \psi_i = \mathbf{rng}\, \partial(\mathbf{p}_{m,n} \circ \Phi^{-1} \circ \phi_i)(t_i) = \mathbf{p}_{m,n}[\mathbf{rng}\, \partial(\Phi^{-1} \circ \phi_i)(t_i)] = \mathbf{R}^m.$$

□

Definition. Suppose V is an open subset of $\mathbf{R}^n$ and $M \in \mathbf{M}_m(V)$. We say ϕ is a **local parameter for** M if there are T and U such that $U \subset V$, $(T, \phi, U) \in \mathbf{Imm}_m$ and

$$U \cap M = \mathbf{rng}\, \phi.$$

We have just shown that if $\phi_i,\ i = 1, 2$ are local parameters for M then $\phi_2^{-1} \circ \phi_1 \in \mathbf{Diffeo}_m$.