Two examples illustrating the Inverse Function Theorem.

Example One. Let $L(x) = x$ for $x \in \mathbf{R}$, $a = 0$,let

$$p(x) = \begin{cases} 0 & \text{if } x \le -1, \\ x+1 & \text{if } -1 < x \le 0, \\ 1-x & \text{if } 0 < x \le 1, \\ 0 & 1 < x. \end{cases}$$

Note that $\mathbf{Lip}\,(p) = 1$ on any interval containing 0 and that

$$\beta = \inf\{|L(x)| : |x| = 1\} = 1.$$

Let $f(x) = L(x) + p(x)$. Since f is constant on $[0,1]$ we see that f is not invertible on any set whose interior contains 0. Thus the hypothesis $\alpha < \beta$ in the Inverse Function Theorem cannot be weakened.

Example Two. Let

$$f : \mathbf{R}^2 \to \mathbf{R}^2$$

be such that $f(x) = (\sin(x_1 + x_2), \exp(x_1) + \exp(-2x_2))$ for $x \in \mathbf{R}^2$. Evidently, $f(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{m}(\partial f(x)) = \begin{bmatrix} \cos(x_1 + x_2) & \cos(x_1 + x_2) \\ \exp(x_1) & -2\exp(-2x_2) \end{bmatrix}.$$

Let $L = \partial f(\mathbf{0})$; evidently,

$$\mathbf{m}(L) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Now $\mathbf{trace}\, L = -1$ and $\mathbf{det}\, L = -3$ so the eigenvalues of L are the roots of

$$0 = \lambda^2 - (\mathbf{trace}\, L)\lambda + \mathbf{det}\, L = \lambda^2 + \lambda - 3$$

which are $\frac{-1 \pm \sqrt{13}}{2}$. Since $\mathbf{m}(L)$ is symmetric we find that

$$\frac{-1 + \sqrt{13}}{2}|x| \le |L(x)| \le \frac{1 + \sqrt{13}}{2}|x| \quad \text{whenever } x \in \mathbf{R}^2$$

and that these inequalities are sharp. Hence

$$||L|| = \frac{1 + \sqrt{13}}{2} \quad \text{and} \quad \beta = \frac{-1 + \sqrt{13}}{2}$$

where we have set

$$\beta = \inf\{|L(x)| : x \in \mathbf{R}^2 \text{ and } |x| = 1\}.$$

Let us set

$$p(x) = f(x) - L(x), \ x \in \mathbf{R}^2.$$

Then

$$\partial p(x) = \partial f(x) - L$$

so

$$\mathbf{m}(\partial p(x)) = \begin{bmatrix} \cos(x_1 + x_2) - 1 & \cos(x_1 + x_2) - 1 \\ \exp(x_1) - 1 & -2(\exp(-2x_2) + 2) \end{bmatrix}.$$

Set

$$\alpha(R) = \mathbf{Lip}\,(p|\mathbf{B_0}(R)), \quad 0 < R < \infty.$$

From earlier work know that

$$\alpha(R) = \sup\{||\partial p(x)|| : x \in \mathbf{B_0}(R)\}.$$

From Taylor's theorem with the Lagrange form for the remainder we have

$$|\cos(t) - 1| \leq |t|, \qquad |\exp(t) - 1| \leq \exp(|t|)|t|, \quad t \in \mathbf{R}.$$

Thus for $x \in \mathbf{B_0}(R)$ the absolute value of the terms in the first row of the matrix of $\partial p(x)$ do not exceed $2R$ and the absolute value of the terms in the second row of the matrix of $\partial p(x)$ do not exceed $2\exp(2R)R$. Consequently,

$$\alpha(R) \leq 4\exp(2R);$$

(We have used here the fact that the $||\cdot||$ norm of a linear transformation from one Euclidean space to another does not exceed either the square root of the dimension of the domain times the maximum length of any column of its standard matrix or the square root of the dimension of its range times the the maximum length of any row of its standard matrix. See below.) Thus the Inverse Function Theorem applies if

$$4\exp(2R)R < \frac{-1+\sqrt{13}}{2}.$$

Life would have been easier had we taken $f(x) = (\sin(x_1 + x_2), \exp(x_1) + \exp(-x_2))$. In this case we have $||L|| = \beta = \sqrt{2}$. But I decided to do an ugly one ...

More stuff on norms. Suppose X and Y are inner product spaces and $L \in \mathbf{B}(X; Y)$. Then

$$||L|| = \sup\{L(x) \bullet y : x \in X,\ |x| \leq 1,\ y \in Y,\ |y| \leq 1\}.$$

Indeed,

$$\begin{aligned} ||L|| &= \sup\{|L(x)| : x \in X,\ |x| \leq 1\} \\ &= \sup\{\sup\{L(x) \bullet y : y \in Y,\ |y| \leq 1\} : x \in X,\ |x| \leq 1\} \\ &= \sup\{L(x) \bullet y : x \in X,\ |x| \leq 1,\ y \in Y,\ |y| \leq 1\}. \end{aligned}$$

Since

$$\begin{aligned} &\sup\{L(x) \bullet y : x \in X,\ |x| \leq 1,\ y \in Y,\ |y| \leq 1\} \\ &\qquad = \sup\{x \bullet L^*(y) : x \in X,\ |x| \leq 1,\ y \in Y,\ |y| \leq 1\} \\ &\qquad = \sup\{L^*(y) \bullet x :\ y \in Y,\ |y| \leq,\ x \in X,\ |x| \leq 1\} \end{aligned}$$

we find that

$$||L|| = ||L^*||.$$

Finally, if X is finite dimensional and $u_1, \ldots, u_n$ is an orthonormal basic sequence for X then

$$\begin{aligned} |L(x)| &= |\sum_{j=1}^{n} (x \bullet u_j) L(u_j)| \\ &\leq \big(\sum_{j=1}^{n} |x \bullet u_j|\big) \max\{|L(u_j) : j \in \{1, \ldots, n\} \\ &\leq \sqrt{n}|x| \max\{|L(u_j) : j \in \{1, \ldots, n\}. \end{aligned}$$

We also have

$$\begin{aligned} |L(x)|^2 &= |\sum_{j=1}^{n} (x \bullet u_j) L(u_j)|^2 \\ &\leq \big(\sum_{j=1}^{n} (x \bullet u_j)^2\big)\big(\sum_{j=1}^{n} |L(u_j)|^2\big) \\ &= |L|^2|x|^2 \end{aligned}$$

where $|L| = \sqrt{\mathbf{trace}\, L^* \circ L}$ is the Euclidean norm of L.