Inversion.

Let

$$\iota : \mathbf{C} \sim \{0\} \to \mathbf{C} \sim \{0\}$$

be **inversion**; that is

$$\iota(z) = \frac{1}{z} \quad \text{whenever } z \in \mathbf{C} \sim \{0\}.$$

Let's give two proofs that

$$\iota'(z) = -\frac{1}{z^2}, \quad z \in \mathbf{C} \sim \{0\}.$$

Suppose $a \in \mathbf{C} \sim \{0\}$. Let

$$g(h) = \frac{1}{a+h} - \frac{1}{a} - \Big(-\frac{h}{a^2}\Big) \quad \text{for } h \in \mathbf{C} \sim \{-a\}.$$

Proof One. Let $\epsilon > 0$ and We have

$$\begin{aligned} |g(h)| &= \Big|\frac{a^2 - a(a+h) + h(a+h))}{a^2(a+h)}\Big| \\ &= \Big|\frac{h^2}{a^2(a+h)}\Big| \\ &\le 2\frac{|h|^2}{|a|^3} \quad \text{provided } |h| \le \tfrac{|a|}{2} \\ &\le \epsilon|h| \quad \text{provided } |h| \le \tfrac{\epsilon|a|^3}{2}. \end{aligned}$$

So

$$g(h) \le \epsilon|h| \quad \text{if} \quad 0 < |h| \le \epsilon = \min\{\frac{|a|}{2}, \epsilon\frac{|a|^3}{2}\}.$$

Proof Two. We have

$$\frac{1}{a+h} = \frac{1}{a}\frac{1}{1-\big(-\frac{h}{a}\big)} = \frac{1}{a}\sum_{m=0}^{\infty}\big(-\frac{h}{a}\big)^m$$

whenever $a + h \in D = \{w \in \mathbf{C} : |w - a| < |a|\}$. Thus

$$g(h) = \frac{h^2}{a^3}\sum_{m=2}^{\infty}\big(-\frac{h}{a}\big)^m.$$

Our assertion now follows from our theory of infinite series.

The second proof is amenable to generalization as follows. Let X be a Banach space and let $\mathcal{I}$ be the set of invertible members of $\mathbf{B}(X; X)$. Thus $A \in \mathcal{I}$ if $A \in \mathbf{B}(X; X)$ and there is $B \in \mathbf{B}(X, X)$ such that AB and BA equal the identity map of X. One easily verifies that such a B is unique and we denote it by

$$A^{-1}.$$

Let

$$\iota(A) = A^{-1} \quad \text{whenever } A \in \mathcal{I}.$$

As an exercise show that $\mathcal{I}$ is open and that ι is differentiable at A and determine its differential at A. Use the above to try to guess what the differential at A is. Here is a big hint as to how to proceed. Suppose $A \in \mathcal{I}$ and $H \in \mathbf{B}(X; X)$ is such that

$$(1) \qquad ||H|| < \frac{1}{||A||}.$$

If $A + H \in \mathcal{I}$ we have

$$(A+H)^{-1} = A^{-1}\big(1 - (-H \circ A^{-1})\big) = A^{-1} \sum_{m=0}^{\infty} (-H \circ A^{-1})^m.$$

Justify this; that is, show that that if (1) holds then the series converges absolutely. Then show its sum is $(A+H)^{-1}$. Finally, show that ι is differentiable at A and determine what it's differential is.

Bonus question. Show that ι has derivatives of all orders.