

1. INTEGRATION THEORY, PART TWO.

1.1. **The Monotone Convergence Theorem.** The following Theorem is fundamental.

**Theorem 1.1. (The Monotone Convergence Theorem.** Suppose  $f$  is a non-decreasing sequence in  $\mathcal{F}_n^+$ . Then

$$(1) \quad \mathbf{I}(\sup_{\nu} f_{\nu}) = \sup_{\nu} \mathbf{I}(f_{\nu}).$$

*Proof.* Let  $a$  and  $b$  be the left and right hand sides of (1), respectively. Owing to the monotonicity of  $\mathbf{I}$ , we find that  $b \leq a$ . Thus we need only show that  $a \leq b$  and we may assume that  $b < \infty$ .

To this end, let  $\epsilon > 0$ . For each  $\nu \in \mathbb{N}$  let  $s_{\nu} \in \mathcal{S}_{n,\uparrow}^+$  be such that  $f_{\nu} \leq \sup s_{\nu}$  and

$$I_{n,\uparrow}^+(s_{\nu}) \leq \mathbf{I}(f_{\nu}) + 2^{-\nu-1}\epsilon.$$

For each  $\mu, \nu \in \mathbb{N}$  with  $\mu \leq \nu$  we let

$$S_{\mu}^{\nu} = \bigvee_{\eta=\mu}^{\nu} s_{\eta} \in I_{n,\uparrow}^+.$$

We define the sequence  $t$  by letting

$$t_{\nu} = (S_0^{\nu})_{\nu} \in \mathcal{S}_n^+.$$

For any  $\nu \in \mathbb{N}$  we have

$$t_{\nu} = (S_0^{\nu})_{\nu} \leq (S_0^{\nu+1})_{\nu} \leq (S_0^{\nu+1})_{\nu+1} = t_{\nu+1}$$

so  $t \in \mathcal{S}_{n,\uparrow}^+$  and

$$(1) \quad I_n^+(t_{\nu}) = I_n^+((S_0^{\nu})_{\nu}) \leq I_{n,\uparrow}^+(S_0^{\nu}).$$

Moreover, for any  $\nu, \xi \in \mathbb{N}$ , we have

$$(s_{\nu})_{\xi} \leq (s_{\nu})_{\nu \vee \xi} \leq (S_0^{\nu \vee \xi})_{\nu \vee \xi} = t_{\nu \vee \xi} \leq \sup t;$$

it follows that  $f_{\nu} \leq \sup t$  for any  $\nu \in \mathbb{N}$  which in turn implies that  $\sup f \leq \sup t$  so

$$\mathbf{I}(\sup f) \leq I_{n,\uparrow}^+(t).$$

We will complete the proof by showing that

$$(2) \quad I_{n,\uparrow}^+(t) \leq \sup_{\nu} \mathbf{I}(f_{\nu}) + \epsilon.$$

Suppose  $\mu, \nu \in \mathbb{N}$  and  $\mu < \nu$ . Since  $s_{\mu} \leq S_{\mu}^{\nu}$  we have

$$f_{\mu} \leq f_{\nu} \wedge f_{\mu+1} \leq (\sup s_{\nu}) \wedge (\sup S_{\mu+1}^{\nu}) = \sup(s_{\nu} \wedge S_{\mu+1}^{\nu}).$$

Using the fact that  $a \wedge b + a \vee b = a + b$  whenever  $a, b \in [0, \infty]$  we find that

$$s_{\mu} \wedge S_{\mu+1}^{\nu} + S_{\mu}^{\nu} = s_{\mu} \wedge S_{\mu+1}^{\nu} + s_{\mu} \vee S_{\mu+1}^{\nu} = s_{\mu} + S_{\mu+1}^{\nu};$$

thus

$$\begin{aligned}
\mathbf{l}(f_\mu) + I_{n,\uparrow}^+(S_\mu^\nu) &\leq I_{n,\uparrow}^+(s_\mu \wedge S_{\mu+1}^n) + I_{n,\uparrow}^+(S_\mu^\nu) \\
&= I_{n,\uparrow}^+(s_\mu \wedge S_{\mu+1}^n + S_\mu^\nu) \\
&= I_{n,\uparrow}^+(s_\mu + S_{\mu+1}^\nu) \\
&= I_{n,\uparrow}^+(s_\mu) + I_{n,\uparrow}^+(S_{\mu+1}^\nu) \\
&\leq \mathbf{l}(f_\mu) + 2^{-\mu-1}\epsilon + I_{n,\uparrow}^+(S_{\mu+1}^\nu).
\end{aligned}$$

Since  $\mathbf{l}(f_\mu) < \infty$  we obtain

$$I_{n,\uparrow}^+(S_\mu^\nu) \leq I_{n,\uparrow}^+(S_{\mu+1}^\nu) + 2^{-\mu-1}\epsilon;$$

Summing from  $\mu = 0$  to  $\nu$  and using (1) we find that

$$I_n^+(t_\nu) \leq I_{n,\uparrow}^+(S_0^\nu) \leq I_{n,\uparrow}^+(S_\nu^\nu) + \epsilon = I_{n,\uparrow}^+(s_\nu) + \epsilon < \mathbf{l}(f_\nu) + \epsilon^{-\nu-1} + \epsilon$$

thereby establishing (2).  $\square$

**Corollary 1.1. ( Fatou's Lemma.)** Suppose  $f$  is a sequence in  $\mathcal{F}_n^+$ . Then

$$\mathbf{l}(\liminf_\nu f_\nu) \leq \liminf_\nu \mathbf{l}(f_\nu).$$

*Proof.* Let  $F_\nu = \inf_{0 \leq \mu \leq \nu} f_\mu$  for each  $\nu \in \mathbb{N}$  and apply the Monotone Convergence Theorem to  $F$ .  $\square$

## 1.2. Basic theory of Lebesgue integration.

**Theorem 1.2.** Suppose  $f \in \mathcal{F}_n^+ \cap \mathbf{Leb}_n$ . Then

$$\mathbf{l}(f) = \mathbf{L}(f).$$

*Proof.* Let  $\epsilon > 0$ . By a previous Proposition there is  $s \in \mathcal{S}_n^+$  such that  $\mathbf{l}(f-s) < \epsilon/2$ . Since  $\mathbf{l}(s) = I_n(s) = \mathbf{L}(s)$  it follows that

$$|\mathbf{l}(f) - \mathbf{L}(f)| \leq |\mathbf{l}(f) - \mathbf{l}(s)| + |\mathbf{L}(f-s)| \leq 2\mathbf{l}(|f-s|) \leq \epsilon.$$

Thus (ii) holds.  $\square$

**Lemma 1.1.** Suppose  $f$  is a sequence in  $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$  such that

(i)  $\sup_\nu f_\nu(x) < \infty$  for each  $x \in \mathbb{R}^n$  and

(ii)  $\mathbf{l}(\sup_\nu f_\nu) < \infty$ .

Then  $\sup_\nu f_\nu \in \mathbf{Leb}_n$ .

*Proof.* Replacing  $f_\nu$  by  $\sup_{0 \leq \mu \leq \nu} f_\mu$  if necessary we may assume without loss of generality that  $f$  is nondecreasing.

Let  $\epsilon > 0$ . Since (ii) holds we may choose  $N \in \mathbb{N}$  such that

$$\sup_\nu \mathbf{l}(f_\nu) \leq \mathbf{l}(f_N) + \epsilon.$$

It follows from the preceding Proposition that

$$\mathbf{l}(f_\nu - f_N) = \mathbf{L}(f_\nu - f_N) = \mathbf{L}(f_\nu) - \mathbf{L}(f_N) = \mathbf{l}(f_\nu) - \mathbf{l}(f_N)$$

for any  $\nu \in \mathbb{N}$  so that

$$\sup_\nu \mathbf{l}(f_\nu - f_N) \leq \epsilon.$$

Since  $f$  is nondecreasing we may use the Monotone Convergence Theorem to infer that

$$\mathbf{l}(\sup_{\nu} f_{\nu} - f_N) = \mathbf{l}(\sup_{\nu} (f_{\nu} - f_N)) = \sup_{\nu} \mathbf{l}(f_{\nu} - f_N) \leq \epsilon.$$

□

**Lemma 1.2.** Suppose  $f$  is a sequence in  $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$ . Then  $\inf_{\nu} f_{\nu} \in \mathbf{Leb}_n$ .

*Proof.* For each  $\nu \in \mathbb{N}$  let  $F_{\nu} = \inf_{0 \leq \mu \leq \nu} f_{\mu} \in \mathbf{Leb}_n$ . Evidently,  $F$  is nonincreasing so  $\mathbb{N} \ni \nu \mapsto F_0 - F_{\nu}$  is nondecreasing. Since

$$\inf_{\nu} F_{\nu} = F_0 - \sup_{\nu} (F_0 - F_{\nu})$$

and since  $\inf_{\nu} f_{\nu} = \inf_{\nu} F_{\nu}$  this Lemma follows from Lemma 1.2. □

**Theorem 1.3.** Suppose  $F \in \mathcal{F}_n^+$ ,  $\mathbf{l}(F) < \infty$  and there is a sequence  $f$  in  $\mathbf{Leb}_n$  such that

$$F(x) = \lim_{\nu \rightarrow \infty} f_{\nu}(x) \quad \text{for } x \in \mathbb{R}^n.$$

Then  $F \in \mathbf{Leb}_n$ .

*Proof.* Choose a  $s \in \mathcal{S}_{n,\uparrow}^+$  such that  $F \leq \sup s$  and  $I_{n,\uparrow}^+(s) < \infty$ . Using the Lemmas 1.1 and 1.2 we infer that, for each  $\xi \in \mathbb{N}$ ,

$$F \wedge s_{\xi} = \inf_{\nu} \sup_{\mu \geq \nu} f_{\mu} \wedge s_{\xi} \in \mathbf{Leb}_n.$$

Since  $F = \sup_{\xi} F \wedge s_{\xi}$  the Theorem follows from the Lemma 1.1. □

**Theorem 1.4. (The Lebesgue Dominated Convergence Theorem.)** Suppose

(i)  $f$  is a sequence in  $\mathbf{Leb}_n$  and  $F \in \mathcal{F}_n$  is such that

$$\lim_{\nu \rightarrow \infty} f_{\nu}(x) = F(x) \quad \text{for all } x \in \mathbb{R}^n;$$

(ii)  $g$  is a sequence in  $\mathbf{Leb}_n$  such that

$$|f_{\nu}| \leq g_{\nu}, \quad \nu \in \mathbb{N};$$

(iii)  $G \in \mathcal{F}_n$ ,

$$\lim_{\nu \rightarrow \infty} g_{\nu}(x) = G(x) \quad \text{for all } x \in \mathbb{R}^n \quad \text{and} \quad \lim_{\nu \rightarrow \infty} \mathbf{l}(g_{\nu}) = \mathbf{l}(G) < \infty.$$

Then  $F \in \mathbf{Leb}_n$  and

$$\lim_{\nu \rightarrow \infty} \mathbf{l}(|F - f_{\nu}|) = 0.$$

*Proof.* For each  $\nu \in \mathbb{N}$  let  $h_{\nu} = G + g_{\nu} - |F - f_{\nu}| \in \mathcal{F}_n^+ \cap \mathbf{Leb}_n$ . We know from the previous Theorem that  $G$  and  $|F - f_{\nu}| = \lim_{\mu \rightarrow \infty} |f_{\mu} - f_{\nu}|$ ,  $\nu \in \mathbb{N}$  are in  $\mathbf{Leb}_n$ . Thus, for any  $\nu \in \mathbb{N}$ ,

$$\mathbf{L}(h_{\nu}) = \mathbf{L}(G) + \mathbf{L}(g_{\nu}) - \mathbf{L}(|F - f_{\nu}|)$$

so

$$\mathbf{l}(h_{\nu}) = \mathbf{l}(G) + \mathbf{l}(g_{\nu}) - \mathbf{l}(|F - f_{\nu}|).$$

By Fatou's Lemma we have

$$2\mathbf{l}(G) = \mathbf{l}(\liminf_{\nu \rightarrow \infty} h_{\nu}) \leq \liminf_{\nu \rightarrow \infty} \mathbf{l}(h_{\nu}).$$

Since

$$\liminf_{\nu \rightarrow \infty} \mathbf{1}(h_\nu) = 2\mathbf{1}(G) - \limsup_{\nu \rightarrow \infty} \mathbf{1}(|F - f_\nu|).$$

it follows that

$$\limsup_{\nu \rightarrow \infty} \mathbf{1}(|F - f_\nu|) = 0.$$

This in turn implies that  $F \in \mathbf{Leb}_n$ .  $\square$

### 1.3. Lebesgue measurable sets.

**Theorem 1.5.** Suppose  $A$  in a nondecreasing sequence of subsets of  $\mathbb{R}^n$ . Then

$$|\cup_{\nu=0}^{\infty} A_\nu|^* = \sup_{\nu} |A_\nu|^*.$$

*Proof.* Since  $\mathbf{1}^+(1_B) = |B|^*$  for any subset  $B$  of  $\mathbb{R}^n$  this follows from the Monotone Convergence Theorem.  $\square$

**Theorem 1.6.** Suppose  $E \subset \mathbb{R}^n$ . The following are equivalent:

- (i)  $1_E \in \mathbf{Leb}_n$ ;
- (ii) for each  $\epsilon > 0$  there is a multirectangle  $M$  such that

$$|(E \sim M) \cup (M \sim E)|^* \leq \epsilon.$$

*Proof.* Suppose (i) holds and  $\epsilon > 0$ . Choose  $s \in \mathcal{S}_n^+$  such that  $\mathbf{1}(|1_E - s|) \leq \epsilon/2$ , let  $M = \{x \in \mathbb{R}^n : s(x) \geq 1/2\}$  and note that  $M$  is a multirectangle. Since

$$\frac{1}{2} \mathbf{1}_{(E \sim M) \cup (M \sim E)} = \frac{1}{2} |1_E - 1_M| \leq |1_E - s|$$

we find that  $|(E \sim M) \cup (M \sim E)|^* \leq \epsilon$ . Thus (ii) holds.

Suppose (ii) holds and  $\epsilon > 0$ . Then there is a multirectangle  $M$  such that  $\mathbf{1}(|1_E - 1_M|) \leq \epsilon$ . Thus  $1_E \in \mathbf{Leb}_n$  since  $1_M \in \mathcal{S}_n$ .  $\square$

**Definition 1.1.** We let

$$\mathbf{Leb}_n^+$$

be the family of functions in  $\mathcal{F}_n^+$  which are the supremum of a nondecreasing sequence  $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$ .

**Theorem 1.7.** Suppose  $F, G \in \mathbf{Leb}_n^+$ . Then

$$\mathbf{1}^+(F + G) = \mathbf{1}^+(F) + \mathbf{1}^+(G).$$

*Proof.* Let  $f, g$  be nondecreasing sequences  $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$  with suprema  $F$  and  $G$ , respectively. Using the Monotone Convergence Theorem three times we calculate

$$\begin{aligned} \mathbf{1}^+(F + G) &= \sup_{\nu} \mathbf{1}^+(f_\nu + g_\nu) \\ &= \sup_{\nu} \mathbf{L}(f_\nu + g_\nu) \\ &= \sup_{\nu} \mathbf{L}(f_\nu) + \mathbf{L}(g_\nu) \\ &= \sup_{\nu} \mathbf{1}^+(f_\nu) + \mathbf{1}^+(g_\nu) \\ &= \mathbf{1}^+(f) + \mathbf{1}^+(g). \end{aligned}$$

$\square$

**Definition 1.2.** We say a subset  $E$  of  $\mathbb{R}^n$  is **Lebesgue measurable** if  $1_E \in \text{Leb}_n^+$  in which case we let

$$\mathcal{L}^n(E) = \mathbf{I}^+(1_E)$$

and call this nonnegative extended real number the **Lebesgue measure of  $E$** .

**Theorem 1.8.** Suppose  $E \subset \mathbb{R}^n$ . Then  $E$  is Lebesgue measurable if for each  $\epsilon > 0$  there is a countable family  $\mathcal{R}$  of rectangles such that

$$|(E \sim (\cup \mathcal{R}) \cup ((\cup \mathcal{R}) \sim E)|^* \leq \epsilon.$$

**Theorem 1.9.** The following statements hold.

- (i)  $\mathcal{M}_n \subset \mathcal{L}_n$ .
- (ii)  $E \in \mathcal{L}_n$  and  $|E|^* < \infty$  if and only if for each  $\epsilon > 0$  there is a multirectangle  $M$  such that

$$|(E \sim M) \cup (M \sim E)|^* \leq \epsilon.$$

- (iii)  $E \in \mathcal{L}_n$  if and only if there is nondecreasing sequence  $F$  in  $\{G \in \mathcal{L}_n : |G|^* < \infty\}$  such that  $E = \cup_{\nu=0}^{\infty} F_{\nu}$ .
- (iv) If  $E, F \in \mathcal{L}_n$  then  $E \cup F, E \cap F, E \sim F \in \mathcal{L}_n$  and

$$|E \cup F|^* + |E \cap F|^* = |E|^* + |F|^*.$$

If  $\mathcal{E}$  is a countable nonempty family of Lebesgue measurable subsets of  $\mathbb{R}^n$  the following assertions hold:

- (v)  $\cup \mathcal{E}$  and  $\cap \mathcal{E}$  are Lebesgue measurable;
- (vi) if  $\mathcal{E}$  is disjointed then

$$|\cup \mathcal{E}|^* = \sum_{E \in \mathcal{E}} |E|^*;$$

- (vii) if  $\mathcal{E}$  is nested then

$$|\cup \mathcal{E}|^* = \sup\{|E|^* : E \in \mathcal{E}\};$$

- (viii) if  $\mathcal{E}$  is nested and  $|E|^* < \infty$  for some  $E \in \mathcal{E}$  then

$$|\cap \mathcal{E}|^* = \inf\{|E|^* : E \in \mathcal{E}\}.$$

*Proof.* Exercise for the reader. □

**Remark 1.1.** In particular, the Lebesgue measurable subsets of  $\mathbb{R}^n$  form a  $\sigma$ -algebra of subsets of  $\mathbb{R}^n$ .

**Definition 1.3.** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . We say  $f$  is **Lebesgue measurable** if  $f^{-1}[U] \in \mathcal{L}_n$  whenever  $U$  is an open subset  $\mathbb{R}$ .

**Proposition 1.1.** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . The following are equivalent.

- (i)  $f$  is Lebesgue measurable.
- (ii)  $\{x \in \mathbb{R}^n : f(x) > c\} \in \mathcal{L}_n$  whenever  $c \in \mathbb{R}$ .
- (iii)  $\{x \in \mathbb{R}^n : f(x) \geq c\} \in \mathcal{L}_n$  whenever  $c \in \mathbb{R}$ .
- (iv)  $\{x \in \mathbb{R}^n : f(x) < c\} \in \mathcal{L}_n$  whenever  $c \in \mathbb{R}$ .
- (v)  $\{x \in \mathbb{R}^n : f(x) \leq c\} \in \mathcal{L}_n$  whenever  $c \in \mathbb{R}$ .

*Proof.* Since

$$\{x \in \mathbb{R}^n : f(x) \geq c\} - \cap \nu = 1^\infty \{x \in \mathbb{R}^n : f(x) > c - \frac{1}{n}\}$$

we see that that (ii) implies (iii). Since

$$\{x \in \mathbb{R}^n : f(x) < c\} = \mathbb{R}^n \sim \{x \in \mathbb{R}^n : f(x) \geq c\}$$

we see that (iii) implies (iv). Since

$$\{x \in \mathbb{R}^n : f(x) \leq c\} = \cap_{\nu=1}^\infty \{x \in \mathbb{R}^n : f(x) < c + \frac{1}{\nu}\}$$

we see that (iv) implies (v). Since

$$\{x \in \mathbb{R}^n : f(x) > c\} = \mathbb{R}^n \sim \{x \in \mathbb{R}^n : f(x) \leq c\}$$

we see that (v) implies (ii). Thus (ii),(iii),(iv) and (v) are equivalent.

(i) obviously implies (ii). Suppose (ii) holds. Then, as (iv) holds,

$$\{x \in \mathbb{R}^n : a < f(x) < b\} \in \mathcal{L}_n \quad \text{whenever } -\infty < a < b < \infty.$$

Let  $U$  be an open subset of  $\mathbb{R}$ . Let  $\mathcal{I}$  be the family of open subintervals of  $U$  with rational endpoints. Then, as  $\mathcal{I}$  is countable, we find that

$$f^{-1}[U] = \cup \{f^{-1}[I] : I \in \mathcal{I}\} \in \mathcal{L}_n$$

. Thus (i) holds. □

**Corollary 1.2.** Suppose  $N$  is a positive integer,  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, N$  are Lebesgue measurable functions, and

$$M : \mathbb{R}^N \rightarrow \mathbb{R}$$

is continuous. Then

$$\mathbb{R}^n \ni x \mapsto M(f_1(x), \dots, f_N(x))$$

is Lebesgue measurable.

**Corollary 1.3.** The set of Lebesgue measurable functions is closed under the arithmetic operation as well as the lattice operations.

**Proposition 1.2.** Suppose  $f$  is a sequence of Lebesgue measurable functions and  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is such that

$$\lim_{\nu \rightarrow \infty} f_\nu(x) = F(x) \quad \text{whenever } x \in \mathbb{R}^n.$$

Then  $F$  is Lebesgue measurable.

*Proof.* Suppose  $c \in \mathbb{R}$ . Then

$$\{x \in \mathbb{R}^n : F(x) > c\} = \cup_{n=1}^\infty \cup_{N=0}^\infty \cap_{\nu=N}^\infty \{x \in \mathbb{R}^n : f_\nu(x) > c + \frac{1}{n}\}.$$

□

**Lemma 1.3.** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $c \in \mathbb{R}$ ,  $E = \{x \in \mathbb{R}^n : f(x) > c\}$  and

$$g_h(x) = \frac{1}{h} [f \wedge (c+h) - f \wedge c] \quad \text{for } h \in (0, \infty).$$

Then

- (i)  $g_h \leq g_k$  if  $0 < k < h < \infty$ ;
- (ii)  $1_E = \sup_{0 < h < \infty} g_h$ .

*Proof.* To prove (i) we suppose  $a \in \mathbb{R}^n$  and  $0 < k < h < \infty$  and we observe that

$$\begin{aligned} f(a) < c &\Rightarrow g_h(a) = 0 = g_k(a), \\ c \leq f(a) < c+k &\Rightarrow g_h(a) = \frac{1}{h}[f(a) - c] \leq \frac{1}{k}[f(a) - c] = g_k(a), \\ c+k \leq f(a) < c+h &\Rightarrow g_h(a) = \frac{1}{h}[f(a) - c] \leq 1 = g_k(a), \\ c+h \leq f(a) &\Rightarrow g_h(a) = 1 = g_k(a). \end{aligned}$$

(ii) is evident.  $\square$

**Lemma 1.4.** Suppose

$$f : \mathbb{R}^n \rightarrow \mathbb{R};$$

$c$  is a sequence of positive real numbers such that

$$\lim_{\nu \rightarrow \infty} c_\nu = 0 \quad \text{and} \quad \sum_{\nu=0}^{\infty} c_\nu = \infty;$$

and  $E$  is the sequence of subsets of  $\mathbb{R}^n$  defined inductively by setting  $E_0 = \{x \in \mathbb{R}^n : f(x) > c_0\}$  and requiring that

$$E_{\nu+1} = \left\{x \in \mathbb{R}^n : f(x) > \sum_{\mu=0}^{\nu} c_\mu 1_{E_\mu}\right\} \quad \text{whenever } \nu > 0.$$

Then

$$f = \sum_{\nu=0}^{\infty} c_\nu 1_{E_\nu}.$$

*Proof.* Straightforward exercise.  $\square$

**Theorem 1.10.** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then  $f \in \mathbf{Leb}_n$  if and only if  $\mathbf{l}(f) < \infty$  and  $f$  is Lebesgue measurable.

*Proof.* Suppose  $f \in \mathbf{Leb}_n$ . Let  $c \in \mathbb{R}$ . That  $\{x \in \mathbb{R}^n : f(x) > c\} \in \mathcal{L}_n$  follows the first of the two preceding Lemmas and our earlier theory.

Suppose  $\mathbf{l}(f) < \infty$  and  $f$  is Lebesgue measurable. Writing  $f = f^+ - f^-$  we see we need only consider the case  $f \geq 0$ . Let  $c$  be a sequence of positive real numbers such that  $\lim_{\nu \rightarrow \infty} c_\nu = 0$  and  $\sum_{\nu=0}^{\infty} c_\nu = \infty$  and let the sequence  $E$  be as in the preceding Lemma so that

$$f = \sum_{\nu=0}^{\infty} c_\nu 1_{E_\nu}.$$

Note that  $E_\nu \in \mathcal{L}_n$ . That  $f \in \mathbf{Leb}_n$  follows from earlier theory.  $\square$

**Theorem 1.11. (The absolute continuity of the integral.)** Suppose  $f \in \mathbf{Leb}_n$ . Then for each  $\epsilon > 0$  there is  $\delta > 0$  such that

$$E \in \mathcal{L}_n \text{ and } |E| < \delta \Rightarrow \mathbf{L}_E(|f|) < \epsilon.$$

*Proof.* For each nonnegative integer  $\nu$  let  $g_\nu = |f| \wedge \nu$ . Since  $g_\nu \uparrow |f|$  as  $\nu \uparrow \infty$  we infer from the Monotone Convergence Theorem that  $\mathbf{l}(g_\nu) \uparrow \mathbf{l}(|f|)$  as  $\nu \uparrow \infty$ . Choose a positive integer  $N$  such that

$$\mathbf{l}(|f|) - \mathbf{l}(g_N) < \frac{\epsilon}{2}.$$

By the preceding theory,  $g_N \in \mathbf{Leb}_n$ . Let  $\delta = \frac{\epsilon}{2N}$ . If  $E \in \mathcal{L}_n$  and  $|E| < \delta$  then

$$|f|1_E = (|f| - g_N)1_E + g_N1_E \leq |f| - g_N + N1_E$$

so that

$$\mathbf{L}_E(|f|) = \mathbf{L}(|f|1_E) \leq \mathbf{L}(|f| - g_N + N1_E) = \mathbf{L}(|f|) - \mathbf{L}(g_N) + N|E| < \epsilon.$$

□

**Theorem 1.12. (Minkowski's inequality in integral form.)** Suppose  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is Lebesgue measurable and  $1 \leq p < \infty$ . Then

$$\left( \int \left| \int h(x, y) dy \right| dx \right)^{1/p} \leq \int \left( \int |h(x, y)|^p dx \right)^{1/p} dy.$$

**Remark 1.2.** We haven't shown that either of the above integrals is defined because we haven't done product integration.

*Proof.* By an approximation argument we need only consider  $h$  of the form

$$h(x, y) = \sum_{j=1}^N f_j(x)1_{F_j}(y), \quad (x, y) \in \mathbb{R} \times \mathbb{R},$$

where  $N$  is a positive integer;  $f_j$  is Lebesgue measurable and  $F_j \in \mathcal{L}_n$ ,  $j = 1, \dots, N$ ; and  $F_i \cap F_j = \emptyset$  if  $1 \leq i < j \leq N$ . We use Minkowski's inequality to estimate

$$\left( \int \left| \int h(x, y) dy \right| dx \right)^{1/p} = \left( \int \left| \sum_{j=1}^N \|F_j\| f_j(x) \right|^p dx \right)^{1/p} \leq \sum_{j=1}^N \|F_j\| \left( \int |f_j(x)|^p dx \right)^{1/p}.$$

But

$$\int \left( \int |h(x, y)|^p dx \right)^{1/p} dy = \sum_{j=1}^N \int_{F_j} \left( \int |h(x, y)|^p dx \right)^{1/p} = \sum_{j=1}^N \int_{F_j} \left( \int |f_j(x)|^p dx \right)^{1/p}.$$

□

**Corollary 1.4.** Suppose  $f$  and  $g$  are Lebesgue measurable. Then

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

*Proof.* Using Minkowski's Inequality in integral form we estimate

$$\begin{aligned} \|f * g\|_p &= \left( \int \left| \int f(x-y)g(y) dy \right|^p dx \right)^{1/p} \\ &\leq \int \left( \int |f(x-y)g(y)|^p dx \right)^{1/p} dy \\ &= \int \left( \int |f(x-y)|^p dx \right)^{1/p} |g(y)| dy \\ &= \int \left( \int |f(x)|^p dx \right)^{1/p} |g(y)| dy \\ &= \|f\|_p \|g\|_1. \end{aligned}$$

□

**Proposition 1.3.** Suppose  $1 \leq p < \infty$ ,  $f$  is Lebesgue measurable and

$$\|f\|_p = \left( \int |f(x)|^p dx \right)^{1/p} < \infty.$$

Then for each  $\epsilon > 0$  there is an elementary function  $s$  such that  $\|f - s\|_p < \epsilon$ .

*Proof.* Let  $\epsilon > 0$ .

For each positive integer  $\nu$  let  $E_\nu = \{x \in \mathbb{R}^n : |f(x)| \leq \nu\}$ . Since  $1_{E_\nu}|f|^p \uparrow |f|^p$  as  $\nu \uparrow \infty$  we infer from the Monotone Convergence Theorem and the additivity of the integral that

$$\int_{E_\nu} |f(x)|^p dx \uparrow \int |f(x)|^p dx \quad \text{as } \nu \uparrow \infty.$$

By the additivity of the integral we infer that

$$\|f - 1_{E_\nu}f\|_p^p = \int_{\mathbb{R}^n \sim E_\nu} |f(x)|^p dx = \int |f(x)|^p dx - \int_{E_\nu} |f(x)|^p dx \downarrow 0 \quad \text{as } \nu \uparrow \infty.$$

We may therefore choose a positive integer  $N$  such that  $\|f - 1_{E_N}f\|_p \leq \epsilon/2$ . Since  $f1_{E_N} \in \mathbf{Leb}_1$  we may choose an elementary function  $s$  such that  $|s| \leq M$  and

$$(2M)^p \int |f1_{E_N} - s|(x) dx \leq \left(\frac{\epsilon}{2}\right)^p.$$

Then

$$\|f1_{E_N} - s\|_p^p = \int |f1_{E_N} - s|^p dx \leq (2M)^p \int |f1_{E_N} - s| dx \leq \left(\frac{\epsilon}{2}\right)^p.$$

It follows from Minkowski's Inequality that

$$\|f - s\|_p \leq \|f - 1_{E_N}f\|_p + \|1_{E_N}f - s\|_p \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

1.3.1. *Smoothing.* Let

$$\phi : \mathbb{R}^n \rightarrow \mathbb{R}$$

be a smooth function such that

- (i)  $0 \leq \phi$ ;
- (ii)  $\mathbf{cl}\{x \in \mathbb{R}^n : \phi(x) \neq 0\} \subset \{x \in \mathbb{R}^n : |x| < 1\}$ ;
- (iii)  $\int \phi(x) dx = 1$ .

For each  $\epsilon > 0$  we let

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n} \phi\left(\frac{x}{\epsilon}\right) \quad \text{for } x \in \mathbb{R}^n.$$

Then

- (i)  $0 \leq \phi_\epsilon$ ;
- (ii)  $\mathbf{cl}\{x \in \mathbb{R}^n : \phi_\epsilon(x) \neq 0\} \subset \{x \in \mathbb{R}^n : |x| < \epsilon\}$ ;
- (iii)  $\int \phi_\epsilon(x) dx = 1$ .

**Theorem 1.13.** Suppose  $1 \leq p < \infty$  and  $f$  is measurable and

$$\int |f(x)|^p dx < \infty.$$

Then  $\phi_\epsilon * f$  is smooth and

$$\|f - \phi_\epsilon * f\|_p \rightarrow 0 \quad \text{as } \epsilon \downarrow 0.$$

*Proof.* Let  $\eta > 0$  and let  $s$  be an elementary function such that  $\|f - s\|_p < \eta/3$ .  
Then

$$\|f - \phi_\epsilon * f\|_p \leq \|f - s\|_p + \|s - \phi * s\|_p + \|\phi_\epsilon * (f - s)\|_p \leq \eta + \|s - \phi * s\|_p.$$

Finally, as  $s$  is elementary,  $\|s - \phi_\epsilon * s\|_p \rightarrow 0$  as  $\epsilon \downarrow 0$ . (Do you see why?)  $\square$