

### Inner products.

Let  $V$  be a vector space.

**Definition.** We say a function

$$\beta : V \times V \rightarrow \mathbf{R}$$

is an **inner product on  $V$**  if

(i) for each  $v \in V$  the function

$$V \ni w \mapsto \beta(v, w) \in \mathbf{R}$$

is linear;

(ii) for each  $w \in V$  the function

$$V \ni v \mapsto \beta(v, w) \in \mathbf{R}$$

is linear;

(iii)  $\beta(v, w) = \beta(w, v)$  for each  $v, w \in V$  and

(iv)  $\beta(v, v) \geq 0$  with equality only if  $v = 0$ .

Here's some fancy mathematics terminology that goes with this. Properties (i) and (ii) say that  $i$  is **bilinear**, property (iii) says that  $i$  is **symmetric** and property (iv) says that  $i$  is **positive definite**.

One often writes

$$v \bullet w$$

for  $\beta(v, w)$ . Keeping in mind (iv), for  $v \in V$  we set

$$|v| = \sqrt{v \bullet v}$$

and call this nonnegative real number the **norm** or **length** of  $v$ .

We have the all important

**Cauchy-Schwarz inequality.** For any  $v, w \in V$  we have

$$|v \bullet w| \leq |v| |w|,$$

with equality only if the set  $\{v, w\}$  is dependent.

**Proof.** We shall assume the  $w \neq 0$  since otherwise the assertion holds trivially. For any  $t \in \mathbf{R}$  we have

$$\begin{aligned} 0 &\leq |v + tw|^2 \\ &= (v + tw) \bullet (v + tw) \\ &= v \bullet v + v \bullet (tw) + (tw) \bullet v + (tw) \bullet (tw) \\ &= v \bullet v + tv \bullet w + tw \bullet v + t^2 w \bullet w \\ &= |v|^2 + 2tv \bullet w + t^2 |w|^2. \end{aligned}$$

Setting

$$t = -\frac{v \bullet w}{|w|^2}$$

and doing a little bit of manipulation we infer the desired inequality. If  $|v \bullet w| = |v| |w|$  we find that  $|v + tw| = 0$  so that  $\{v, w\}$  is indeed dependent.  $\square$

**The triangle inequality.** For any  $v, w \in V$  we have that

$$|v + w| \leq |v| + |w|,$$

with equality only if  $\{v, w\}$  is dependent.

**Proof.** Square both sides and use the Cauchy-Schwartz inequality.  $\square$

**The Parallelogram Law.** Suppose  $v, w \in V$ . Then

$$|v + w|^2 + |v - w|^2 = 2(|v|^2 + |w|^2).$$

**Proof.** Turn the crank.  $\square$

Things to do.

i) Justify the use of term *length* above.

ii) We say  $v$  is **perpendicular to**  $w$  and write

$$v \perp w$$

if

$$v \bullet w = 0.$$

Justify the use of this terminology.

iii) Suppose  $f$  and  $g$  are continuous real value functions on the interval  $[a, b]$ . Show that

$$\left| \int_a^b f(x)g(x) dx \right| \leq \left( \int_a^b f(x)^2 dx \right)^{\frac{1}{2}} \left( \int_a^b g(x)^2 dx \right)^{\frac{1}{2}}.$$

## Hilbert space

Let  $A$  be a set.

We let

$$\mathbf{H}_A = \{x \in \mathbf{R}^A : \sum_{\alpha \in A} x_\alpha^2 < \infty\}.$$

Suppose  $x, y \in \mathbf{H}_A$  and  $F$  is a finite subset of  $A$ . By the Cauchy-Schwarz inequality in  $\mathbf{R}^F$  we have

$$\left( \sum_{\alpha \in F} |x_\alpha y_\alpha| \right)^2 \leq \left( \sum_{\alpha \in F} x_\alpha^2 \right) \left( \sum_{\alpha \in F} y_\alpha^2 \right) \leq \left( \sum_{\alpha \in \alpha} x_\alpha^2 \right) \left( \sum_{\alpha \in A} y_\alpha^2 \right) < \infty.$$

Thus we may define

$$x \bullet y = \sum_{\alpha \in A} x_\alpha y_\alpha$$

and thereby make  $\mathbf{H}_A$  into an inner product space. We have

$$|x|^2 = \sum_{\alpha \in A} x_\alpha^2 \text{ for } x \in \mathbf{H}_A.$$

For each  $\alpha \in A$  we define  $\mathbf{e}_\alpha \in \mathbf{H}_A$  by setting

$$(\mathbf{e}_\alpha)_\beta = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{else,} \end{cases}$$

we note the  $|\mathbf{e}_\alpha| = 1$  and we define the linear map

$$\mathbf{e}^\alpha : \mathbf{H}_A \rightarrow \mathbf{R}$$

by setting  $e^\alpha(x) = x_\alpha$ . Evidently,

$$|e^\alpha(x)| \leq |x|, \quad x \in \mathbf{H}_A.$$

For each subset  $B$  of  $A$  we define the linear map

$$P_B : \mathbf{H}_A \rightarrow \mathbf{H}_A$$

by setting

$$P_B(x)_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in B, \\ 0 & \text{else} \end{cases}$$

for  $x \in \mathbf{H}_A$ . We have

$$|P_{B \cup C}(x)|^2 + |P_{B \cap C}(x)|^2 = |P_B(x)|^2 + |P_C(x)|^2 \quad \text{whenever } B, C \subset A \text{ and } x \in \mathbf{H}_A.$$

**Theorem.**  $\mathbf{H}_A$  is complete.

**Proof.** Suppose  $\mathcal{C}$  is a nonempty family of nonempty closed subsets of  $\mathbf{H}_A$  such that

$$(1) \quad \inf\{\mathbf{diam} C : C \in \mathcal{C}\} = 0.$$

For each  $\alpha \in A$  let  $\mathcal{C}_\alpha = \{\mathbf{cl} e^\alpha[C] : C \in \mathcal{C}\}$ . Inasmuch as

$$\mathbf{diam} \mathbf{cl} e^\alpha[C] \leq \mathbf{diam} C \quad \text{whenever } C \in \mathcal{C}$$

we infer that  $\mathcal{C}_\alpha$  is a nonempty family of nonempty closed subsets of  $\mathbf{R}$  such  $\inf\{\mathbf{diam} C : C \in \mathcal{C}_\alpha\} = 0$  whenever  $\alpha \in A$ . Owing to the completeness of  $\mathbf{R}$  we may define  $x \in \mathbf{R}^A$  by requiring that

$$x_\alpha \in \bigcap \{\mathbf{cl} e^\alpha[C] : C \in \mathcal{C}\}, \quad \alpha \in A.$$

Note that

$$(2) \quad |x_\alpha - y_\alpha| \leq \mathbf{diam} C \quad \text{whenever } y \in C \in \mathcal{C}.$$

To prove the Theorem we will show that

$$(3) \quad x \in \bigcap \mathcal{C}.$$

**Lemma One.**  $x \in \mathbf{H}_A$ .

**Proof.** Suppose  $F$  is a finite subset of  $A$ . Using the Cauchy-Schwartz Inequality in  $\mathbf{R}^F$  we find that, for any  $y \in C \in \mathcal{C}$ ,

$$|x|_F \leq |(x - y)|_F + |y|_F = \sqrt{\sum_{\alpha \in F} (x_\alpha - y_\alpha)^2} + |P_F(y)| \leq \sqrt{|F| \mathbf{diam} C} + |y|.$$

That  $x \in \mathbf{H}_A$  now follows from (1).  $\square$

**Lemma Two.** Suppose  $C \in \mathcal{C}$  and  $\epsilon > 0$ . There exists a finite subset  $G$  of  $A$  such that

$$|P_{A \sim G}(y)| \leq \mathbf{diam} C + \epsilon \quad \text{for } y \in C.$$

**Proof.** Let  $z \in C$ . Let  $G$  be a finite subset of  $A$  such that  $|P_{A \sim G}(z)| < \epsilon$ . For any  $y \in C$  we have

$$|P_{A \sim G}(y)| \leq |P_{A \sim G}(y - z)| + |P_{A \sim G}(z)| \leq |y - z| + |P_{A \sim G}(z)| \leq \mathbf{diam} C + \epsilon$$

□

Suppose  $C \in \mathcal{C}$  and  $\epsilon > 0$ . (3) will follow if we can show that

$$(4) \quad \mathbf{B}_x(\epsilon) \cap C \neq \emptyset.$$

Using Lemma Two and (1) choose a finite subset  $G$  of  $A$  and  $D \in \mathcal{C}$  such that  $D \subset C$  and  $|P_{A \sim G}(y)| \leq \epsilon/3$  whenever  $y \in D$ . Next use Lemma One to choose a finite subset  $F$  of  $A$  such that  $G \subset F$  and  $|P_{A \sim F}(x)| \leq \epsilon/3$ . Finally use (1) to choose  $E \in \mathcal{C}$  such that  $E \subset D$  and  $\sqrt{|F|} \mathbf{diam} E \leq \epsilon/3$ . Let  $y \in D$ . Then by (2)

$$|x - y| \leq |P_F(x - y)| + |P_{A \sim F}(x)| + |P_{A \sim F}(y)| \leq \sqrt{|F|} \mathbf{diam} E + |P_{A \sim F}(x)| + |P_{A \sim G}(y)| \leq \epsilon$$

so  $y \in \mathbf{B}_x(\epsilon)$  and (4) holds. □

Let

$$\mathbf{B}_A = \{x \in \mathbf{H}_A : |x| \leq 1\}.$$

Then  $\mathbf{B}_A$  is closed and bounded. It is *not* totally bounded if  $A$  is infinite. That is because

$$|\mathbf{e}_\alpha - \mathbf{e}_\beta| = \sqrt{2} \text{ whenever } \alpha, \beta \in A \text{ and } \alpha \neq \beta$$

so that if  $0 < r < \sqrt{2}$  then no two of the  $\mathbf{e}_\alpha$ 's can belong to any closed ball of radius  $r$  and so  $\mathbf{H}_A$  is not contained in the union of a *finite* number of closed balls of radius  $r$ .