1. Fourier series.

Definition 1.1. Given a real number P, we say a complex valued function f on $\mathbb{R}$ is **P-periodic** if
$$f(x+P) = f(x) \quad \text{for all } x \in \mathbb{R}.$$
We let
$$\mathcal{P}$$
be the set of complex valued 2π-periodic functions f on $\mathbb{R}$ such that
$$1_I f \in \mathbf{Leb}_1 \quad \text{whenever } I \text{ is a bounded interval.}$$
(Replace $\mathbf{Leb}_1$ by $\mathbf{Riem}_1$ if $\mathbf{Leb}_1$ makes you nervous. A great deal of what follows will still go through.) It follows from our previous work that $\mathcal{P}$ is a vector space over $\mathbb{C}$ with respect to pointwise addition and scalar multiplication.

Here is a Corollary of Hölder's Inequality.

Theorem 1.1. Suppose $1 \le p < q \le \infty$. Then
$$||f||_p \le (2\pi)^{1/p-1/q}||f||_q \quad \text{whenever } f \in \mathcal{P}.$$
In particular,
$$\mathcal{P}_q \subset \mathcal{P}_p.$$

Proof. If $q = \infty$ the inequality holds trivially (Why?) so suppose $q < \infty$. Let $\tilde{p} = q/p$ and $\tilde{q} = \tilde{p}/(\tilde{p}-1)$ so $\tilde{p}$ and $\tilde{q}$ are conjugate. From the Hölder's Inequality we infer that
$$||f||_p^p = |||f|^p 1_{\mathbb{R}}||_1 \le |||f|^p||_{\tilde{p}}||1_{\mathbb{R}}||_{\tilde{q}} = ||f||_q^p (2\pi)^{1-p/q}.$$
□

Of particular interest is the case $p = 2$; we will frequently write
$$||f||$$
for $||f||_2$.

Proposition 1.1. For any $f \in \mathcal{P}_1$ and any $a \in \mathbb{R}$ we have
$$\int_{-\pi+a}^{\pi+a} f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx.$$

Exercise 1.1. Prove this Proposition. To start, show that if f is Riemann or Lebesgue integrable on $\mathbb{R}^n$ then
$$\mathbf{I}(\tau_a f) = \mathbf{I}(f) \quad \text{for } a \in \mathbb{R}^n$$
where $\mathbf{I}$ is either $\mathbf{R}$ or $\mathbf{L}$, respectively. Here $\tau_a(x) = x + a$ for $x \in \mathbb{R}^n$ and $\tau_a f = f \circ (\tau_a)^{-1} = f \circ \tau_{-a}$.

Definition 1.2. Suppose $f, g \in \mathcal{P}$. We say f and g are **complementary** if $fg \in \mathcal{P}_1$ in which case we set
$$(f,g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$
If $f \in \mathcal{P}_p$ and $g \in \mathcal{P}_q$ for some p, q such that $1 \le p, q \le \infty$ and p and q are conjugate then f and g are complementary by virtue of Hölders Inequality.

In case $p = 2 = q$ one easily verifies from the linearity of the integral that

$$(\cdot, \cdot)$$

is a pseudo-Hermitian inner product on $\mathcal{P}_2$. The "pseudo" is necessary because

$$||f||_2 = \sqrt{(f, f)} = 0$$

only implies that $\{x \in \mathbb{R} : f(x) \neq 0\}$ is a set of measure zero and not that $f = 0$ which means, by definition that $f(x) = 0$ for all $x \in \mathbb{R}$. (Note that if $0 < \eta < \infty$ then

$$\eta^2 \mathcal{L}^1(\{x \in (-\pi, \pi] : |f(x)| \geq \eta\}) \leq \int_{-\pi}^{\pi} |f(x)|^2 \, dx = ||f||^2.)$$

Definition 1.3. For each $n \in \mathbb{Z}$ we let

$$E_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \quad \text{for } x \in \mathbb{R};$$

evidently, $E_n \in \mathcal{P}_\infty$.

1.1. **Discussion and more definitions.** Suppose $A \in \mathbb{C} \sim \{0\}$. Then $\frac{d}{dx} \frac{e^{Ax}}{A} = e^{Ax}$, $x \in \mathbb{R}$. So if $-\infty < a < b < \infty$ we may use the Fundamental Theorem of Calculus to obtain

$$\int_a^b e^{Ax} \, dx = \frac{e^{Ax}}{A} \Big|_a^b = \frac{e^{Ab} - e^{Aa}}{A}.$$

Thus

$$(E_m, E_n) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{else.} \end{cases}$$

That is, the set $\{E_n : n \in \mathbb{Z}\}$ is **orthonormal** with respect to $(\cdot, \cdot)$.

For each $N \in \mathbb{N}$ we let

$$\mathcal{T}_N$$

be the linear subspace of $\mathcal{P}_\infty$ spanned by $\{E_n : |n| \leq N\}$ and we call the members of $\mathcal{T}_N$ **trigonometric polynomials of degree** N**.**

For each $f \in \mathcal{P}_1$ we define

$$\hat{f} : \mathbb{Z} \to \mathbb{C},$$

the **Fourier transform of** f, by letting

$$\hat{f}(n) = (f, E_n) \quad \text{for } n \in \mathbb{Z}.$$

One of our goals is to reconstruct f *from its Fourier transform.* As a first step in this direction, for each nonnegative integer N and each $f \in \mathcal{P}_1$ we set

$$S_N f \; = \sum_{|n| \leq N} (f, E_n) E_n \; = \sum_{|n| \leq N} \hat{f}(n) E_n.$$

In particular, if $f \in \mathcal{P}_2$ then $S_N f$ is the orthogonal projection with respect to $(\cdot, \cdot)$ of f onto $\mathcal{T}_N$.

Theorem 1.2. (Bessel's Inequality.) For any $f \in \mathcal{P}_1$ we have

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \; \leq \; ||f||^2.$$

Remark 1.1. Plancherel's Theorem, which comes later, will give the opposite inequality.

Proof. This follows from our work on orthogonal projections.

Here is a recap of what we did there. Let N be a positive integer N. Keeping in mind the orthogonality of the E_n's we obtain

$$\begin{aligned}
0 &\le (f - \sum_{n=-N}^{N} \hat{f}(n)E_n, f - \sum_{n=-N}^{N} \hat{f}(n)E_n) \\
&= (f,f) - \sum_{|n|\le N} (\hat{f}(n)E_n, f) - \sum_{|n|\le N} (f, \hat{f}(n)E_n) + (\sum_{|n|\le N} \hat{f}(n)E_n, \sum_{|n|\le N} \hat{f}(n)E_n) \\
&= (f,f) - \sum_{|n|\le N} \hat{f}(n)(E_n, f) - \sum_{|n|\le N} \overline{\hat{f}(n)}(f, E_n) + \sum_{|n|\le N} \hat{f}(n)\overline{\hat{f}(n)}E_n \\
&= ||f||^2 - \sum_{|n|\le N} |\hat{f}(n)|^2.
\end{aligned}$$

□

The Fourier transform behaves nicely with respect to translations. Suppose $h \in \mathbb{R}$ and $f \in \mathcal{P}$. Recall that

$$\tau_h f(x) = f(x-h) \quad \text{for } x \in \mathbb{R}.$$

By the by the translation invariance of integration and the first Proposition in this section we have

$$\int_{-\pi}^{\pi} \tau_h |f(x)|\,dx = \int_{-\pi-h}^{\pi-h} |f(x)|\,dx = \int_{-\pi}^{\pi} |f(x)|\,dx < \infty$$

so that $\tau_h f \in \mathcal{P}$. It follows that $\tau_h f \in \mathcal{P}_p$ whenever $1 \le p \le \infty$ and $f \in \mathcal{P}_p$.

Proposition 1.2. We have

(i) τ_h is linear for each $h \in \mathbb{R}$;
(ii) $\tau_{h_1} \circ \tau_{h_2} = \tau_{h_1+h_2}$ whenever $h_1,\ h_2 \in \mathbb{R}$;
(iii) $(\tau_h f, \tau_h g) = (f,g)$ whenever $f, g \in \mathcal{P}$ and $h \in \mathbb{R}$.

Exercise 1.2. Prove this. It's very straightforward.

Proposition 1.3. Suppose $f \in \mathcal{P}$ and $h \in \mathbb{R}$. Then

$$\widehat{\tau_h f}(n) = e^{-inh}\hat{f}(n).$$

Exercise 1.3. Exercise.

Definition 1.4. Suppose $f, g \in \mathcal{P}_1$. For each $x \in \mathbb{R}$ we set

$$f * g(x) = \begin{cases} \int_{-\pi}^{\pi} f(x-y)g(y)\,dy & \text{if } \int_{-\pi}^{\pi} |f(x-y)g(y)|\,dy < \infty \\ 0 & \text{else} \end{cases}$$

and we call $f * g$ the **convolution** of f and g.

Recalling Tonelli's Theorem we note that $f * g$ is a complex valued 2π-periodic Lebesgue measurable function on $\mathbb{R}$. Recalling Young's Inequality we note that if $p, q, r \in [1, \infty]$,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

$f \in \mathcal{P}_p$ and $g \in \mathcal{P}_q$ then

$$||f * g||_r \le ||f||_p ||g||_q.$$

By Tonelli's Theorem the first of the above cases holds for almost all $x \in \mathbb{R}$. Evidently, $f * g$ is 2π-periodic. Our previously developed theory of convolutions, with natural modifications, applies here.

Proposition 1.4. Suppose $f, g \in \mathcal{P}$. Then

$$\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}.$$

Exercise 1.4. Prove this.

Definition 1.5. For each nonnegative integer N we define the **Dirichlet kernel** D_N by letting

$$D_N = \frac{1}{\sqrt{2\pi}} \sum_{|n| \leq N} E_n.$$

Proposition 1.5. Let N be a nonnegative integer. Then

(i)
$$D_N(x) = \frac{1}{2\pi} \begin{cases} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} & \text{if } x \neq 0, \\ 2N+1 & \text{else;} \end{cases}$$

(ii) D_N is even and $\int_{-\pi}^{\pi} D_N(x)\,dx = 1$ and

(iii) $S_N f = D_N * f$ for any $f \in \mathcal{P}_1$.

Proof. Suppose N and $x \in \mathbb{R} \sim \{0\}$. Then

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} \\ &= \frac{1}{2\pi} e^{-iNx} \frac{1-(e^{ix})^{2N+1}}{1-e^{ix}} \\ &= \frac{1}{2\pi} \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} \\ &= \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}, \end{aligned}$$

and it is evident that $S_N(0) = \frac{2N+1}{2\pi}$ so (i) holds.

To prove (ii), note that

$$\int_{-\pi}^{\pi} e^{inx}\,dx = 2\pi(E_n, E_0) = \begin{cases} 0 & \text{if } n \neq 0, \\ 2\pi & \text{else} \end{cases}$$

for any integer n.

To prove (iii), suppose $f \in \mathcal{P}$ and $x \in \mathbb{R}$ and observe that

$$\begin{aligned} S_N f(x) &= \sum_{|n| \leq N} (f, E_n) E_n \\ &= \frac{1}{2\pi} \sum_{|n| \leq N} \left(\int_{-\pi}^{\pi} f(t) e^{-int}\,dt \right) e^{inx} \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{|n| \leq N} e^{in(x-t)} \right) f(t)\,dt \\ &= D_N * f(x). \end{aligned}$$

$\square$

Lemma 1.1. (The Riemann Lebesgue Lemma.) Suppose f is Lebesgue integrable on $\mathbb{R}$. (This means that $f \in \mathbf{Leb}_1$.) Then

$$\lim_{t\to\infty} \int f(x) \sin tx \, dx \;=\; 0.$$

Proof. Suppose $\infty < a < b < \infty$. Then

$$\int_a^b \sin tx \, dx = \frac{\cos tb - \cos ta}{t} \;\to\; 0 \;\text{ as }\; t \;\to\; \infty.$$

It follows that

$$\int s(x) \, \sin tx \, dx \;\to\; 0 \quad \text{as} \quad t \;\to\; \infty. \tag{1}$$

for any elementary function s such that $\{x \in \mathbb{R} : s(x) \neq 0\}$ is bounded..

Let $\eta > 0$. Choose an elementary function s such that $\{x \in \mathbb{R} : s(x) \neq 0\}$ is bounded and

$$\int |f - s| \leq \eta;$$

(Remember this is practically the definition of what it means for f to be Lebesgue integrable.) Then

$$\begin{aligned}
\left| \int f(x) \sin tx \, dx \right| &= \left| \int [f(x) - s(x)] \sin tx \, dx + \int s(x) \sin tx \, dx \right| \\
&\leq \int |f(x) - s(x)| \, dx + \left| \int s(x) \sin tx \, dx \right| \\
&\leq \eta + \left| \int s(x) \sin tx \, dx \right|;
\end{aligned}$$

for any $t \in \mathbb{R}$. Now use (1) to complete the proof. $\square$

Corollary 1.1. Suppose $f \in \mathcal{P}$. Then

$$\lim_{|n|\to\infty} \hat{f}(n) \;=\; 0.$$

Theorem 1.3. Suppose $f \in \mathcal{P}_1$, $x \in \mathbb{R}$, $L^+, L^- \in \mathbb{C}$ and

$$\lim_{\delta\downarrow 0} \left(\int_{(x-\delta,x)} \left| \frac{f(t) - L^-}{t - x} \right| dt + \int_{(x,x+\delta)} \left| \frac{f(t) - L^+}{t - x} \right| dt \right) = 0. \tag{1}$$

Then

$$\lim_{N\uparrow\infty} S_N f(x) \;=\; \frac{L^- + L^+}{2}.$$

Remark 1.2. (Very important) For example, if f is differentiable at x the hypothesis holds with $L^\pm = f(x)$.

Proof. **Part One.** Suppose $x = 0$. Set $K^- = [-\pi, 0)$ and $K^+ = (0, \pi)$. For each positive integer N let

$$g_N^\pm(t) \;= D_N(-t)(f(t) - L^\pm) \quad \text{for } t \in K^\pm.$$

Recall that D_N is even and $\int_{-\pi}^{\pi} D_N = 1$; this implies

$$\int_{K^-} D_N \;=\; \frac{1}{2} \;=\; \int_{K^+} D_N.$$

. Thus

$$\begin{aligned} S_N f(0) - \frac{L^- + L^+}{2} & \\ &= \int_{-\pi}^{\pi} D_N(-t) f(t)\, dt - L^- \int_{-\pi}^{0} D_N(-t)\, dt - L^+ \int_0^{\pi} D_N(-t)\, dt \\ &= \int_{K^-} g_N^-(t)\, dt + \int_{K^+} g_N^+(t)\, dt. \end{aligned}$$

For any $\delta \in (0, \pi)$ we set

$$J_\delta^\pm \;=\; K^\pm \cap (-\delta, \delta), \quad I_\delta^\pm \;=\; K^\pm \sim J_\delta^\pm.$$

Let $\eta > 0$. Since

$$|g_N^\pm(t)| \;\le\; \left| \frac{f(t) - L^\pm}{-t} \right| \left| \frac{-t}{\sin \frac{-t}{2}} \right|$$

for any $t \in (-\pi, \pi)$ and any N we may choose $\delta \in (0, \pi)$ such that

$$\int_{J_\delta^\pm} |g_N^\pm| \;\le\; \eta.$$

Moreover,

$$I_\delta^\pm \ni t \mapsto \frac{f(t) - L^\pm}{\sin \frac{-t}{2}}$$

is Lebesgue integrable so, by the Riemann-Lebesgue Lemma,

$$\lim_{N \uparrow \infty} \int_{I_\delta^\pm} g_N^\pm \;=\; 0;$$

the Theorem follows in case $x = 0$.

Part Two. From Part One we infer that

$$\lim_{N \to \infty} S_N(\tau_{-x} f)(0) = \frac{L^- + L^+}{2}.$$

But

$$S_N(\tau_{-x} f)(0) = \frac{1}{\sqrt{2\pi}} \sum_{|n| \le N} \widehat{\tau_{-x} f}(n) = \frac{1}{\sqrt{2\pi}} \sum_{|n| \le N} \hat{f}(n) e^{inx} = S_N f(x).$$

$\square$

Proposition 1.6. Suppose m is a positive integer, $f \in \mathcal{P}$ and f is m times continuously differentiable. Then

$$\widehat{f^{(m)}}(n) = (in)^m \hat{f}(n).$$

Proof. We use integration by parts to obtain

$$\begin{aligned}
\sqrt{2\pi}\hat{f'}(n) &= \int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx \\
&= \int_{-\pi}^{\pi} e^{-inx}\, d(f(x)) \\
&= e^{-inx}\, f(x)|_{x=-\pi}^{x=\pi} - \int_{-\pi}^{\pi} f(x)\, d(e^{-inx}) \\
&= in \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \\
&= \sqrt{2\pi}\, in\, \hat{f}(n).
\end{aligned}$$

Thus the Proposition holds if $m = 1$ and follows for arbitrary m by induction. □

Corollary 1.2. Suppose m is a positive integer, $f \in \mathcal{P}$ and f is m times continuously differentiable, N is a nonnegative integer and $x \in \mathbb{R}$. Then

$$|f(x) - S_N f(x)| \le \frac{1}{\sqrt{\pi}} \frac{1}{N^{\frac{2m-1}{2}}} ||f^{(m)}||.$$

Proof. Note that

$$\sum_{n\in\mathbb{Z}} |a_n b_n| \le \left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{1/2} \left(\sum_{n\in\mathbb{Z}} |b_n|^2\right)^{1/2}$$

whenever a and b are complex valued functions on $\mathbb{Z}$; that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^{2m}} \le \int_N^{\infty} \frac{dx}{x^{2m}} = \frac{1}{N^{2m-1}};$$

and that, by Bessel's Inequality,

$$\sum_{|n|>N} |\widehat{f^{(m)}}(n)|^2 \le ||f^{(m)}||^2.$$

Let $O \in \mathbb{N}$ and $x \in \mathbb{R}$. From the Fourier Inversion Formula we have that

$$f(x) - S_N f(x) = \lim_{O\to\infty} \sum_{N<n\le O} \hat{f}(n) E_n(x).$$

Thus if $O \in \mathbb{N}$ and $N < O$ we have

$$\begin{aligned}
\left|\sum_{N<|n|<O} \hat{f}(n) E_n(x)\right| &= \left|\sum_{N<|n|<O} \frac{1}{(in)^m} \widehat{f^{(m)}}(n) E_n(x)\right| \\
&\le \frac{1}{\sqrt{2\pi}} \left(\sum_{N<|n|<O} \frac{1}{n^{2m}}\right)^{1/2} \left(\sum_{|n|>N} |\widehat{f^{(m)}}(n)|^2\right)^{1/2} \\
&\le \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{N^{2m-1}}}\, ||f^{(m)}||.
\end{aligned}$$

□

1.2. **Approximate identies.** Let ϕ be a smooth nonnegative real valued function on $\mathbb{R}$ which has integral one and which is supported in $[-1,1]$. For each $\epsilon > 0$ let $\phi_\epsilon(x) = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon})$ for $x \in \mathbb{R}$.

Theorem 1.4. Suppose $f \in \mathcal{P}_2$. Then

$$\lim_{N\uparrow\infty} ||f - S_N f|| = 0.$$

Proof. Let $\eta > 0$. Choose $\epsilon > 0$ such that if $g = \phi_\epsilon * f$ then $||f - g|| < \eta$. Then for any nonnegative integer N we have

$$||f - S_N f|| \leq ||f - g|| + ||S_N(f-g)|| + ||S_N g - g|| \leq 2\eta + ||S_N g - g||$$

because $||S_N(f-g)|| \leq ||f - g||$ by Bessel's Inequality. But $||S_N g - g|| \to 0$ as $N \uparrow \infty$ because, by the previous Theorem, $S_N g$ converges uniformly to g as $N \uparrow \infty$ since g is smooth. □

Corollary 1.3. (Plancherel's (or is it Parseval's?) Theorem.) If $f \in \mathcal{P}_2$ then

$$||f|| = \left(\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2\right)^{1/2}.$$

Proof. This follows from the preceding Corollary and the Pythagorean identity

$$||f||^2 = ||S_N f||^2 + ||S_N f - f||^2.$$

□

Corollary 1.4.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Apply Parseval's Theorem to the the member f of $\mathcal{P}$ such that $f(x) = x$ for $x \in [-\pi, \pi)$. □

Corollary 1.5. Suppose $f \in \mathcal{P}$ and $\hat{f} = 0$. Then f equals zero almost everywhere.

Proof. We have $\widehat{\phi_\epsilon * f} = 0$ for any $\epsilon > 0$ so, as $\phi_\epsilon * f$ is smooth, $\phi_\epsilon * f = 0$ by the previous Theorem. Since $||f - \phi_\epsilon * f||_1 \to 0$ as $\epsilon \downarrow 0$ we infer that $f = 0$. □

Corollary 1.6. Suppose $f \in \mathcal{P}$. Then

$$||f|| = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

Proof. In view of Bessel's Inequality and the foregoing we need to show that if the right hand side is finite so is the left hand side, so suppose the right hand side is finite. Now $\widehat{\phi_\epsilon * f} = \widehat{\phi_\epsilon}\hat{f}$ and $|\hat{\phi_\epsilon}| \leq 1$ so $||\phi_\epsilon * f||$ is bounded independently of ϵ by Plancherel's Theorem. Do you know what to do now? □

Definition 1.6. (The Fejer kernel.) For each nonnegative integer N let

$$F_N = \frac{1}{N+1}\sum_{n=0}^{N} D_N.$$

Proposition 1.7.

$$F_N(x) \;=\; \begin{cases} \left(\frac{\sin(\frac{N+1}{2})x}{\sin\frac{x}{2}}\right)^2 & \text{if } x \neq 0, \\ N+1 & \text{else.} \end{cases}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{N}\sum_{|m|\le n} e^{inx} \;&=\; \frac{1}{1-e^{ix}}\sum_{n=0}^{N} e^{-inx} - e^{i(n+1)x} \\
&=\; \frac{1}{1-e^{ix}}\Big[\frac{1-e^{-i(N+1)x}}{1-e^{-ix}} - e^{ix}1-e^{i(N+1)x}over1-e^{ix}\Big] \\
&=\; \frac{1}{1-e^{ix}}\Big[\frac{(1-e^{ix})(1-e^{-i(N+1)x})-(1-e^{-ix})e^{ix}(1-e^{i(N+1)x})}{(1-e^{-ix})(1-e^{ix})}\Big] \\
&=\; \frac{(1-e^{-i(N+1)x})+(1-e^{i(N+1)x})}{(1-e^{-ix})(1-e^{ix})} \\
&=\; \frac{\left(e^{i(\frac{N+1}{2})x}-e^{-i(\frac{N+1}{2})x}\right)^2}{\left(e^{i\frac{x}{2}}-e^{-\frac{x}{2}}\right)^2} \\
&=\; \left(\frac{\sin(\frac{N+1}{2})x}{\sin\frac{x}{2}}\right)^2.
\end{aligned}$$

$\square$