

1. THE FOURIER TRANSFORM ON  $\mathbb{R}$ .

Suppose

$$f : \mathbb{R} \rightarrow \mathbb{C}$$

is integrable; this means, by definition,  $f \in \mathbf{Leb}_1$ . For each  $\xi \in \mathbb{R}$  we set

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi} dx.$$

Evidently,

$$(0) \quad |\hat{f}(\xi)| \leq \int |f(x)| dx \quad \text{whenever } x \in \mathbb{R}.$$

This function is called the **Fourier transform of  $f$** . Note that

$$(1) \quad \lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = 0$$

by the Riemann-Lebesgue Lemma.

**Definition 1.1.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{C}$ . We let

$$\mathbf{spt} f = \mathbf{cl} \{f \neq 0\}$$

and call this closed set the **support of  $f$** . We say  $f$  is **smooth** if  $\mathbf{dmn} f^{(m)} = \mathbb{R}$  for all  $m \in \mathbb{N}$ .

We let

$$\mathcal{D}(\mathbb{R})$$

be the vector space of smooth  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that  $f$  is smooth and has compact support.

**Theorem 1.1.** Suppose  $f \in \mathcal{D}(\mathbb{R})$ . Then

$$\hat{f}'(\xi) = i\xi \hat{f}(\xi) \quad \text{for } \xi \in \mathbb{R}.$$

**Exercise 1.1.** Prove this. Integrate by parts. Because  $f$  has compact support you may work on a bounded open interval which contains the support of  $f$ .

**Corollary 1.1.** Suppose  $f \in \mathcal{D}(\mathbb{R})$ . Then

$$(2) \quad |\hat{f}(\xi)| \leq \frac{1}{|\xi|^m} \int |f^{(m)}(x)| dx \quad \text{for all } m \in \mathbb{N} \text{ and all } \xi \in \mathbb{R} \setminus \{0\}.$$

*Proof.* This is an immediate consequence of the preceding Theorem. □

**Corollary 1.2.** Suppose  $f \in \mathcal{D}(\mathbb{R})$ . Then

$$\int |\hat{f}(\xi)| d\xi < \infty.$$

**Exercise 1.2.** Prove this.

1.1. **The Fourier Inversion Formula (Smooth version.)** Suppose

$$f \in \mathcal{D}(\mathbb{R}).$$

Then

$$(3) \quad f(x) = \frac{1}{2\pi} \int \hat{f}(\xi) e^{ix\xi} d\xi \quad \text{for } x \in \mathbb{R}.$$

The remainder of this subsection is devoted to a sketch of a proof of (3). We will use the Fourier Inversion Formula for periodic functions. The details are left as exercises for the reader.

Let  $M$  be a positive integer such that

$$\text{spt } f \subset (-M, M).$$

Let

$$\mathbf{M} = \{m \in \mathbb{N} : m \geq M\}.$$

For each  $m \in \mathbf{M}$  let  $g_m : \mathbb{R} \rightarrow \mathbb{C}$  be determined by the requirement that

$$g_m(t) = f\left(\frac{mt}{\pi}\right) \quad \text{whenever } -\pi < t < \pi$$

and that  $g_m$  is  $2\pi$ -periodic. Note that  $g_m$  is smooth.

**Proposition 1.1.** We have

$$\widehat{g_m}(n) = \frac{1}{\sqrt{2\pi}} \frac{\pi}{m} \hat{f}\left(\frac{n\pi}{m}\right) \quad \text{whenever } m \in \mathbf{M} \text{ and } n \in \mathbb{Z}.$$

**Exercise 1.3.** Prove this. Turn the crank.

Suppose  $x \in \mathbb{R}$ ,  $m \in \mathbf{M}$  and  $N$  is a positive integer. From the Fourier Inversion Formula in the periodic case we obtain

$$\begin{aligned} f(x) &= g_m\left(\frac{\pi x}{m}\right) \\ &= \sum_{n \in \mathbb{Z}} \widehat{g_m}(n) E_n\left(\frac{\pi x}{m}\right) \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{\pi}{m} \hat{f}\left(\frac{n\pi}{m}\right) e^{\frac{in\pi x}{m}} \\ &= \frac{1}{2\pi} (T_1(x, m, N) + T_2(x, m, N) + T_3(x, m, N) + T_4(x)). \end{aligned}$$

where we have set

$$\begin{aligned} T_1(x, m, N) &= \sum_{n=-mN}^{mN-1} \int_{\frac{n\pi}{m}}^{\frac{(n+1)\pi}{m}} \left( \hat{f}\left(\frac{n\pi}{m}\right) e^{\frac{in\pi x}{m}} - \hat{f}(\xi) e^{ix\xi} \right) d\xi, \\ T_2(x, m, N) &= \left( \sum_{n < -mN} + \sum_{n \geq mN} \right) \frac{\pi}{m} \hat{f}\left(\frac{n\pi}{m}\right) e^{\frac{in\pi x}{m}}, \\ T_3(x, N) &= - \int_{|\xi| > M} \hat{f}(\xi) e^{ix\xi} d\xi, \\ T_4(x) &= \int \hat{f}(\xi) e^{ix\xi} d\xi. \end{aligned}$$

Let  $\epsilon > 0$ .

**Exercise 1.4.** Show that for any  $N \in \mathbb{N}^+$  and any  $\epsilon > 0$  there is  $M \in \mathbf{M}$  such that

$$|T_1(x, m, N)| \leq \epsilon \quad \text{whenever } x \in \mathbb{R} \text{ and } m \geq M.$$

Hint: Note that

$$\mathbb{R} \ni \xi \mapsto \hat{f}(\xi)e^{ix\xi}$$

is uniformly continuous.

**Exercise 1.5.** Show that for any  $\epsilon > 0$  there is  $O \in \mathbb{N}^+$  such that

$$(4) \quad |T_2(x, m, N)| \leq \epsilon \quad \text{whenever any } x \in \mathbb{R}, m \in \mathbf{M} \text{ and } N \geq O.$$

Hint: Use (2).

**Exercise 1.6.** Show that for any  $\epsilon > 0$  there is  $O \in \mathbb{N}^+$  such that

$$|T_3(x, N)| \leq \epsilon \quad \text{whenever } x \in \mathbb{R} \text{ and } N \geq O.$$

Hint: Use (2).

**Exercise 1.7.** Combine all these estimates we see that (3) holds.

**Theorem 1.2.** Suppose  $f, g \in \mathcal{D}(\mathbb{R})$ . We have

$$\int f(x)\overline{g(x)} dx = \frac{1}{2\pi} \int \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi$$

*Proof.* Using the Fourier Inversion Formula we calculate

$$\begin{aligned} 2\pi \int f(x)\overline{g(x)} dx &= \int f(x) \left( \int e^{ix\xi} \hat{g}(\xi) d\xi \right) dx \\ &= \int f(x) \left( \int e^{-ix\xi} \overline{\hat{g}(\xi)} d\xi \right) dx \\ &= \int \left( \int e^{-ix\xi} f(x) dx \right) \overline{\hat{g}(\xi)} d\xi \\ &= \int \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi. \end{aligned}$$

□

**Corollary 1.3. (Plancherel's Theorem.)** Suppose  $f \in \mathcal{D}(\mathbb{R})$ . Then

$$\int |f(x)|^2 dx = \frac{1}{2\pi} \int |\hat{f}(\xi)|^2 d\xi.$$

**Remark 1.1.** The preceding result extends to square summable functions by approximation.

1.2. **The Gaussian.** Let

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{for } x \in \mathbb{R}.$$

Then

$$\begin{aligned} \sqrt{2\pi} \left( \frac{d}{d\xi} + \xi \right) \hat{g}(\xi) &= \int e^{-ix\xi} e^{-x^2/2} (-ix + \xi) dx \\ &= -i \int e^{-(ix\xi + x^2/2)} d_x (ix\xi + x^2/2) \\ &= 0. \end{aligned}$$

**Exercise 1.8.** Justify the steps in the preceding calculation.

Thus

$$\frac{d}{d\xi}\hat{g}(\xi) = -\xi\hat{g}(\xi).$$

It follows that

$$\hat{g}(\xi) = \hat{g}(0)e^{-\xi^2/2} = \sqrt{2\pi}\hat{g}(0)g(\xi).$$

**Exercise 1.9.** Justify the preceding assertion. It follows from uniqueness theorems from ordinary differential equations but you can and should do it directly using material we have already developed.

But

$$\int |g(x)|^2 dx = \frac{1}{2\pi} \int |\hat{g}(\xi)|^2 d\xi = |\hat{g}(0)|^2 \int |g(\xi)|^2 d\xi$$

from which it follows that

$$1 = \hat{g}(0) = \int g(x)$$

so that

$$\hat{g} = \sqrt{2\pi}g.$$

**1.3. Another approach.** You may find this a lot easier. (So why didn't he do this first? Because he wanted to emphasize the link with Fourier series.)

Suppose  $f, g \in \mathcal{D}(\mathbb{R})$ .

**Exercise 1.10.** Show that

$$\int f(x)\hat{g}(x) dx = \int \hat{f}(\xi)g(\xi) d\xi.$$

**Exercise 1.11.** Use the preceding Exercise to show that

$$\int f(\epsilon x)\hat{g}(x) = \int \hat{f}(\eta)g(\epsilon\eta) d\eta \quad \text{for } \epsilon > 0.$$

**Exercise 1.12.** Let  $\epsilon \downarrow 0$  in the preceding Exercise to obtain

$$f(0) \int \hat{g}(x) dx = g(0) \int \hat{f}(\eta) d\eta.$$

**Exercise 1.13.** Use an approximation argument to show that the result of the previous Exercise holds if  $f, g \in \mathbf{Leb}_1$  and  $f, g$  are continuous at 0.

**Exercise 1.14.** Let  $g(x) = e^{-|x|}$  for  $x \in \mathbb{R}$ . Calculate

$$\hat{g}(\xi) = \frac{2}{1+\xi^2} \quad \text{for } \xi \in \mathbb{R}.$$

Voilà! We have another proof of the Fourier inversion formula.