## 1. More on the exponential function.

**Proposition 1.1.** We have

(i) $\exp(\overline{z}) = \overline{\exp(z)}$ for $z \in \mathbf{C}$;
(ii) $|\exp(iy)| = 1$ for $y \in \mathbf{R}$.

*Proof.* Exercise for the reader. □

**Definition 1.1.** We define

$$\cos : \mathbf{C} \to \mathbf{C} \quad \text{and} \quad \sin : \mathbf{C} \to \mathbf{C}$$

by letting

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

whenever $z \in \mathbf{C}$. The reader may want to derive the addition laws for cos and sin from the addition law for the exponential map. Note that $\exp|\mathbf{R}$, $\cos|\mathbf{R}$ and $\sin|\mathbf{R}$ are all real valued. Note that cos is even and that sin is odd.

**Theorem 1.1.** We have

(i) $\exp' = \exp$, $\cos' = -\sin$, and $\sin' = \cos$;
(ii) $\cos(z)^2 + \sin(z)^2 = 1$ whenever $z \in \mathbf{C}$.

*Proof.* We have already shown that $\exp' = \exp$. We leave verification of the remaining assertions in this Theorem as an exercise for the reader; one needs a very weak version of the Chain Rule. □

**Theorem 1.2.** The following statements hold:

(i) $\exp|\mathbf{R}$ is increasing with range equal $(0, \infty)$;
(ii) there is a positive real number

$$\pi$$

such that

$$\{z \in \mathbf{C} : \exp(z) = 1\} = \{2\pi n i : n \in \mathbf{Z}\};$$

(iii) $\exp|\mathbf{R}i$ has range equal $\{z \in \mathbf{C} : |z| = 1\}$;
(iv) $\mathbf{rng}\,\exp = \mathbf{C} \sim \{0\}$.

*Proof.* **Part One.** If $x > 0$ and $h > 0$ then $\exp(x) \geq 1 + x > 1$ and $\exp(x + h) = \exp(x)\exp(h) \geq \exp(x)(1 + h) > \exp(x)$; Since $\exp(0) = 1$ we conclude that $\exp|[0, \infty)$ is increasing with range a subset of $[1, \infty)$. Since $\exp(x) > 1 + x$ whenever $x > 0$ we infer that $\lim_{x\to\infty} \exp(x) = \infty$. It follows from the Intermediate Value Theorem that the range of $\exp|[0, \infty)$ equals $[1, \infty)$. Since $\exp(-x)\exp(x) = \exp(-x + x) = 1$ for any $x \geq 0$ we infer that $\exp|(-\infty, 0]$ is increasing with range equal $(0, 1]$. Thus (1) holds.

Let $T = \{t \in (0, \infty) : \cos(t) = 0\}$.

**Part Two.** I claim that $T$ is nonempty.

Were $T$ empty we could infer from the Intermediate Value Theorem and the fact that $\cos(0) = 1$ that $\cos(t) > 0$ whenever $t \in (0, \infty)$. Since $\sin' = \cos$ it would follow from the Mean Value Theorem that $\sin|[0,\infty)$ is increasing. Let $\eta > 0$. Since $\sin(0) = 0$ we would have

$$0 < \sin(\eta) < \sin(t) \quad \text{whenever } \eta < t < \infty.$$

This would imply

$$\frac{\cos(t) - \cos(\eta)}{t - \eta} < -\sin(\eta) \quad \text{whenever } \eta < t < \infty.$$

because, if $\eta < t < \infty$, the Mean Value Theorem in conjunction with $\cos' = -\sin$ provides $\xi \in (\eta, t)$ such that

$$\frac{\cos(t) - \cos(\eta)}{t - \eta} = -\sin(\xi).$$

But this forces $\cos(t) \to -\infty$ as $t \uparrow \infty$ which is incompatible with $|\cos(t)| \leq 1$ for $t \in \mathbf{R}$. Thus $T$ is nonempty.

**Part Two.** Since cos is continuous and $\cos(0) = 1$, $T$ is a closed set of positive real numbers. Let $P$ be its least element. By the Mean Value Theorem, the Intermediate Value Theorem and the fact that $\sin' = \cos$ we find that $\sin|[0,P]$ is increasing with range $[0,1]$. By this fact, the Mean Value Theorem, the Intermediate Value Theorem and the fact that $\cos' = \sin$ we find that $\cos|[0,P]$ in decreasing with range $[0,1]$. It follows that that $\exp|[0,P]i$ is univalent with range $C = \{u + iv : u \in [0,1],\ v \in [0,1] \text{ and } u^2 + v^2 = 1\}$. Using the fundamental property of the exponential map we infer that $\exp|[P,2P]i$ is univalent with range $iC$; that $\exp|[2P,3P]i$ is univalent with range $(-1)C$; and that $\exp|[3P,4P]i$ is univalent with range $(-i)C$. (ii) now follows with $\pi = 2P$; (iii) follows from (ii) and the fundamental property of the exponential map; (iv) follows from (i), (iii) and the fundamental property of the exponential map. □