1. Hölder's Inequality and Minkowski's inequality.

We fix $p, q \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Definition 1.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable. We let

$$||f||_p = \left(\int |f(x)|^p \, dx \right)^{1/p} \quad \text{if } p < \infty$$

and we let

$$||f||_\infty = \sup\{t \in (0, \infty) : \mathcal{L}^n(\{|f| > t\}) > 0\}.$$

Proposition 1.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable and $c \in \mathbb{R}$. Then

$$||cf||_p = |c| ||f||_p.$$

Proof. Exercise for the reader. □

Theorem 1.1. (Hölder's Inequality.) Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable. Then

$$||fg||_1 \leq ||f||_p ||g||_q.$$

Proof. Exercise for the reader. Here are some hints. Treat the case $p = \infty$ or $q = \infty$ by a straightforward argument. When $p < \infty$ and $q < \infty$ first reduce to the case $||f||_p = 1$ and $||g||_q = 1$ by making use of the previous Proposition; then apply the inequality

$$a^{1/p} b^{1/q} \leq \frac{1}{p} a + \frac{1}{q} b \quad \text{for } a, b \in (0, \infty).$$

□

Theorem 1.2. Minkowski's Inequality. Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable. Then

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

Proof. Exercise for the reader. Here are some hints. The cases $p = 1$ and $p = \infty$ follow from the triangle inequality. In case $1 < p < \infty$ apply Hölder's Inequality to $|f + g|^p \leq |f + g|^{p-1}(|f| + |g|)$. □

1.1. **An extension of Hölder's Inequality.**

Suppose $p, q, r \in [0, \infty]$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Theorem 1.3. Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable. Then

$$||fg||_r \leq ||f||_p ||g||_q.$$

Proof. Exercise for the reader. It easily reduces to the Hölder Inequality. □

1.2. **Minkowski's inequality in integral form.**

Theorem 1.4. (Minkowski's inequality in integral form.) Suppose $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $1 \le p < \infty$. Then

$$\left(\int \left| \int h(x,y)\,dy \right| \,dx \right)^{1/p} \le \int \left(\int |h(x,y)|^p\,dx \right)^{1/p} \,dy.$$

Proof. By an approximation argument we need only consider h of the form

$$h(x,y) = \sum_{j=1}^{N} f_j(x) 1_{F_j}(y), \quad (x,y) \in \mathbb{R} \times \mathbb{R},$$

where N is a positive integer; f_j is Lebesgue measurable; and $F_j \in \mathcal{L}_n$, $j = 1, \ldots, N$; and $F_i \cap F_j = \emptyset$ if $1 \le i < j \le N$. We use Minkowski's inequality to estimate

$$\left(\int \left| \int h(x,y)\,dy \right| \,dx \right)^{1/p} = \left(\left| \sum_{j=1}^{N} ||F_j|| f_j(x) \right|^p \,dx \right)^{1/p} \le \sum_{j=1}^{N} ||F_j|| \left(\int |f_j(x)|^p\,dx \right)^{1/p}.$$

But

$$\int \left(\int |h(x,y)|^p\,dx \right)^{1/p} \,dy = \sum_{j=1}^{N} \int_{F_j} \left(\int |h(x,y)|^p\,dx \right)^{1/p} = \sum_{j=1}^{N} \int_{F_j} \left(\int |f_j(x)|^p\,dx \right)^{1/p}.$$

□

2. Convolution and other stuff.

Definition 2.1. Whenever X is a set, f is a function with domain X and P is a permutation of X we let

$$Pf = f \circ P^{-1};$$

thus if Q is another permutation of X then

$$(P \circ Q)f = P(Qf).$$

Definition 2.2. For each $a \in \mathbb{R}^n$ let

$$\tau_a : \mathbb{R}^n \to \mathbb{R}^n,$$

translation by a, be such that

$$\tau_a(x) = x + a.$$

Evidently,

$$\tau_a^{-1} = \tau_{-a}$$

and

$$\tau_a \circ \tau_b = \tau_{a+b} \quad \text{for } b \in \mathbb{R}^n.$$

Proposition 2.1. Suppose $a \in \mathbb{R}^n$.

If $f \in \mathcal{F}_n^+$ then

$$f \in \mathbf{Leb}_n^+ \iff \tau_a f \in \mathbf{Leb}^+$$

in which case $\mathbf{l}(f) = \mathbf{l}(\tau_a f)$.

If $f \in \mathcal{F}_n$ then

$$f \in \mathbf{Leb}_n \iff \tau_a f \in \mathbf{Leb}$$

in which case $\mathbf{L}(f) = \mathbf{L}(\tau_a f)$.

Proof. Exercise for the reader. Proceed as follows. First, show that $||\tau_a[R]|| = ||R||$ whenever R is a rectangle. Second, show that $I_n(\tau_a s) = I_n(s)$ whenever $s \in \mathcal{S}_n$ and that $I_n^+(\tau_a s) = I_n^+(s)$ whenever $s \in \mathcal{S}_n^+$. Lastly, approximate f above by $s \in \mathcal{S}_n^+$ if $f \in \mathbf{Leb}_n^+$ and by $s \in \mathcal{S}_n$ if $f \in \mathbf{Leb}_n$. □

Definition 2.3. Whenever $f, g \in \mathbf{Leb}_n^+$ we define

$$f * g \in \mathbf{Leb}_n$$

by letting

$$f * g(x) = \int f(x-y)g(y)\,dy \quad \text{for } x \in \mathbb{R}^n.$$

We say the pair $(f,g) \in \mathcal{F}_n \times \mathcal{F}_n$ of functions on $\mathbb{R}^n$ is **admissible** if f and g are Lebesgue measurable and

$$\mathcal{L}^n(\{|f| * |g| = \infty\}) = 0$$

in which case we define

$$f * g : \mathbb{R}^n \to \mathbb{R}$$

by letting

$$f * g(x) = \begin{cases} \int f(x-y)g(y)\,dy & \text{if } |f| * |g|(x) < \infty, \\ 0 & \text{else.} \end{cases}$$

Proposition 2.2. Suppose $f, g \in \mathbf{Leb}_n^+$. Then the following statements hold.

(ii) $f * g = g * f$;
(iii) $f * (g * h) = (f * g) * h$;
(iv) if $a \in \mathbb{R}^n$ then

$$(\tau_a f) * g = \tau_a(f * g) = f * (\tau_a g).$$

Proof. Exercise for the reader. Use Tonelli's Theorem and Proposition 2.1. □

Proposition 2.3. Suppose $f, g, h \in \mathbf{Leb}_n$. Then the following statements hold.

(i) (f,g) is admissible;
(ii) $f * g \in \mathbf{Leb}_n$ and $f * g = g * f$ almost everywhere;
(iii) $f * (g * h) = (f * g) * h$ almost everywhere;
(iv) if $a \in \mathbb{R}^n$ then

$$(\tau_a f) * g = \tau_a(f * g) = f * (\tau_a g) \quad \text{almost everywhere.}$$

Proof. Exercise for the reader. Use Tonelli's Theorem and Proposition 2.1 to show that (f,g) is admissible. You could also peak at the proof of the next Theorem. □

Theorem 2.1. Suppose f and g are Lebesgue measurable, $||f||_p < \infty$ and $||g||_1 < \infty$. Then (f,g) is admissible and

$$||f * g||_p \leq ||f||_p ||g||_1.$$

Proof. Using Minkowski's Inequality in integral form we estimate

$$\begin{aligned}
||f * g||_p &= \left(\int \left| \int f(x-y) g(y)\, dy \right|^p dx \right)^{1/p} \\
&\le \int \left(\int |f(x-y) g(y)|^p \, dx \right)^{1/p} dy \\
&= \int \left(\int |f(x-y)|^p \, dx \right)^{1/p} |g(y)|\, dy \\
&= \int \left(\int |f(x)|^p \, dx \right)^{1/p} |g(y)|\, dy \\
&= ||f||_p ||g||_1.
\end{aligned}$$

□

Proposition 2.4. Suppose $1 \le p < \infty$, f is Lebesgue measurable and

$$||f||_p < \infty.$$

Then for each $\epsilon > 0$ there is an elementary function s such that $||f - s||_p < \epsilon$.

Proof. Let $\epsilon > 0$.

For each positive integer ν let $E_\nu = \{x \in \mathbb{R}^n : |f(x)| \le \nu\}$. Since $1_{E_\nu}|f|^p \uparrow |f|^p$ as $\nu \uparrow \infty$ we infer from the Monotone Convergence Theorem and the additivity of the integral that

$$\int_{E_\nu} |f(x)|^p \, dx \uparrow \int |f(x)|^p \, dx \quad \text{as } \nu \uparrow \infty.$$

By the additivity of the integral we infer that

$$||f - 1_{E_\nu} f||_p^p = \int_{\mathbb{R}^n \sim E_\nu} |f(x)|^p \, dx = \int |f(x)|^p \, dx - \int_{E_\nu} |f(x)|^p \, dx \downarrow 0 \quad \text{as } \nu \uparrow \infty.$$

We may therefore choose a positive integer N such that $||f - 1_{E_N} f||_p \le \epsilon/2$. Since $f 1_{E_N} \in \mathbf{Leb}_1$ we may choose an elementary function s such that $|s| \le N$ and

$$(2N)^p \int |f 1_{E_N} - s|(x)\, dx \le \left(\frac{\epsilon}{2}\right)^p.$$

Then

$$||f 1_{E_N} - s||_p^p = \int |f 1_{E_N} - s|^p \, dx \le (2M)^p \int |f 1_{E_N} - s| \, dx \le \left(\frac{\epsilon}{2}\right)^p.$$

It follows from Minkowski's Inequality that

$$||f - s||_p \le ||f - 1_{E_N} f||_p + ||1_{E_N} f - s||_p \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

2.1. **Smoothing.** Let

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

be a smooth function such that

(i) $0 \le \phi$;
(ii) $\mathbf{cl}\,\{\phi \ne 0\} \subset \{x \in \mathbb{R}^n : |x| < 1\}$;
(iii) $\int \phi(x)\, dx = 1$.

For each $\epsilon > 0$ we let

$$\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\epsilon^{-1}x\right) \quad \text{for } x \in \mathbb{R}^n.$$

Then

(i) $0 \leq \phi_\epsilon$;
(ii) $\mathbf{cl}\,\{x \in \mathbb{R}^n : \phi_\epsilon(x) \neq 0\} \subset \{x \in \mathbb{R}^n : |x| < \epsilon\}$;
s(iii) $\int \phi_\epsilon(x)\,dx = 1$.

Theorem 2.2. Suppose $1 \leq p < \infty$ and f is measurable and

$$\int |f(x)|^p\,dx < \infty.$$

Then $\phi_\epsilon * f$ is smooth and

$$||f - \phi_\epsilon * f||_p \to 0 \text{ as } \epsilon \downarrow 0.$$

Proof. Let $\eta > 0$ and let s be a elementary function such that $||f - s||_p < \eta/3$. Then

$$||f - \phi_\epsilon * f||_p \leq ||f - s||_p + ||s - \phi * s||_p + ||\phi_\epsilon * (f - s)||_p < 2\eta + ||s - \phi * s||_p.$$

Finally, as s is elementary, $||s - \phi_\epsilon * s||_p \to 0$ as $\epsilon \downarrow 0$. (Do you see why?) □