

1. THE RIEMANN AND LEBESGUE INTEGRALS.

Fix a positive integer  $n$ . Recall that

$$\mathcal{R}_n \quad \text{and} \quad \mathcal{M}_n$$

are the family of rectangles in  $\mathbb{R}^n$  and the algebra of multirectangles in  $\mathbb{R}^n$ , respectively.

**Definition 1.1.** We let

$$\mathcal{F}_n^+, \quad \mathcal{F}_n, \quad \mathcal{B}_n,$$

be the set of  $[0, \infty]$  valued functions on  $\mathbb{R}^n$ ; the vector space of real valued functions on  $\mathbb{R}^n$ ; the vector space of  $f \in \mathcal{F}_n$  such that  $\{f \neq 0\} \cup \text{rng } f$  is bounded, respectively.

We let

$$\mathcal{S}_n = \mathcal{B}_n \cap \mathcal{S}(\mathcal{M}_n) \subset \mathcal{F}_n \quad \text{and we let} \quad \mathcal{S}_n^+ = \mathcal{S}_n \cap \mathcal{F}_n^+ \subset \mathcal{S}^+(\mathcal{M}_n).$$

Thus  $s \in \mathcal{S}_n$  if and only if  $s : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\text{rng } s$  is finite,  $\{s = y\}$  is a multirectangle for each  $y \in \mathbb{R}$  and  $\{s \neq 0\}$  is bounded and  $s \in \mathcal{S}_n^+$  if and only if  $s \in \mathcal{S}_n$  and  $s \geq 0$ .

We let  $\mathcal{S}_{n,\uparrow}^+$  be the set of nondecreasing sequences in  $\mathcal{S}_n^+$ . For each  $s \in \mathcal{S}_{n,\uparrow}^+$  we let

$$\sup s \in \mathcal{F}_n^+$$

be such that

$$\sup s(x) = \sup\{s_\nu(x) : \nu \in \mathbb{N}\} \quad \text{for } x \in \mathbb{R}^n$$

and we let

$$I_{n,\uparrow}^n(s) = \sup\{I_n^+(s_\nu) : \nu \in \mathbb{N}^+\}.$$

**Remark 1.1.** We shall prove below the nontrivial Theorem that if  $s, t \in \mathcal{S}_{n,\uparrow}^+$  and  $\sup s = \sup t$  then  $I_{n,\uparrow}^n(s) = I_{n,\uparrow}^n(t)$ .

**Proposition 1.1.** Suppose  $c \in [0, \infty]$  and  $s, t \in \mathcal{S}_{n,\uparrow}^+$ . Then

- (i)  $cs \in \mathcal{S}_{n,\uparrow}^+$  and  $I_{n,\uparrow}^n(cs) = cI_{n,\uparrow}^n(s)$ ;
- (ii)  $s + t \in \mathcal{S}_{n,\uparrow}^+$  and  $I_{n,\uparrow}^n(s + t) = I_{n,\uparrow}^n(s) + I_{n,\uparrow}^n(t)$ ;
- (iii) if  $s \leq t$  then  $I_{n,\uparrow}^n(s) \leq I_{n,\uparrow}^n(t)$ .

*Proof.* Straightforward exercise for the reader. □

**Definition 1.2.** For each  $f \in \mathcal{F}_n$  we let

$$\mathbf{r}(f) = \inf\{I_n^+(s) : s \in \mathcal{S}_n^+ \text{ and } |f| \leq s\}$$

and we let

$$\mathbf{I}(f) = \inf\{I_{n,\uparrow}^n(s) : s \in \mathcal{S}_{n,\uparrow}^+ \text{ and } |f| \leq \sup s\}.$$

For each  $f \in \mathcal{F}_n^+$  we let

$$\mathbf{r}^+(f) = \inf\{I_n^+(s) : s \in \mathcal{S}_n^+ \text{ and } f \leq s\}$$

and we let

$$\mathbf{I}^+(f) = \inf\{I_{n,\uparrow}^n(s) : s \in \mathcal{S}_{n,\uparrow}^+ \text{ and } f \leq \sup s\}.$$

**Remark 1.2.** Evidently,  $\mathbf{r}^+(f) = \mathbf{r}(|f|)$  and  $\mathbf{I}^+(f) = \mathbf{I}(|f|)$  for any  $f \in \mathcal{F}_n$ .

Note also that if  $f \in \mathcal{F}_n$ ,  $g \in \mathcal{B}_n$  and  $|f| \leq g$  then  $f \in \mathcal{B}_n$ . It follows that if  $f \in \mathcal{F}_n$  and  $\mathbf{r}(f) < \infty$  then  $f \in \mathcal{B}_n$ .

**Proposition 1.2.** We have

$$\mathbf{l} \leq \mathbf{r}.$$

*Proof.* Suppose  $f \in \mathcal{F}_n$ ,  $s \in \mathcal{S}_n^+$  and  $|f| \leq s$ . Let  $u$  be the sequence in  $\mathcal{S}_n^+$  whose range equals  $s$ ; that is,  $u_\nu = s$  for all  $\nu \in \mathbb{N}$ . Then  $\sup u = s$  so

$$\mathbf{l}(f) \leq I_{n,\uparrow}^n(u) = I_n^+(s)$$

which is to say  $\mathbf{l}(f)$  is a lower bound for the set of  $I_n^+(t)$  corresponding to  $t \in \mathcal{S}_n^+$  with  $|f| \leq t$ .  $\square$

**Proposition 1.3.**  $\mathbf{r}$  and  $\mathbf{l}$  are extended seminorms on  $\mathcal{F}_n$ .

*Proof.* Straightforward exercise for the reader.  $\square$

**Example 1.1.** Let

$$Q = (0, 1) \cap \mathbb{Q}.$$

We will show that

$$\mathbf{r}(1_Q) = 1 \quad \text{and that} \quad \mathbf{l}(1_Q) = 0.$$

Since  $1_Q \leq 1_{(0,1)} \in \mathcal{S}_n^+$  we find that

$$\mathbf{r}(1_Q) \leq I_n^+(1_{(0,1)}) = \|(0, 1)\| = 1.$$

Suppose  $1_Q \leq s \in I_n^+$ . Let  $y \in [0, \infty]$ . Obviously,

$$s(x) = s(q) \geq 1 \quad \text{whenever } x \in s^{-1}[\{y\}] \text{ and } q \in (0, 1) \cap \mathbb{Q} \cap s^{-1}[\{y\}].$$

It follows that

$$y \geq 1 \quad \text{whenever } (0, 1) \cap \mathbf{int} s^{-1}[\{y\}] \neq \emptyset$$

since, in this case,  $Q \cap s^{-1}[\{y\}] \neq \emptyset$ .

Therefore,

$$\begin{aligned} I_n^+(s) &= \sum_{y \in \mathbf{rng} s} y \|s^{-1}[\{y\}]\| \\ &= \sum_{y \in \mathbf{rng} s} y \|\mathbf{int} s^{-1}[\{y\}]\| \\ &\geq \sum_{y \in \mathbf{rng} s} y \|(0, 1) \cap \mathbf{int} s^{-1}[\{y\}]\| \\ &\geq \sum_{y \in \mathbf{rng} s} \|(0, 1) \cap \mathbf{int} s^{-1}[\{y\}]\| \\ &= \sum_{y \in \mathbf{rng} s} \|(0, 1) \cap s^{-1}[\{y\}]\| \\ &= 1. \end{aligned}$$

Thus

$$\mathbf{r}(1_Q) \geq 1.$$

Let  $q : \mathbb{N} \rightarrow \mathbb{Q} \cap [0, 1]$  be univalent with range  $\mathbb{Q} \cap [0, 1]$ . For each  $\nu \in \mathbb{N}$  let

$$s_\nu = \sum_{\mu=0}^{\nu} 1_{\{q_\mu\}} \in \mathcal{S}_n^+.$$

Note that  $s \in \mathcal{S}_{n,\uparrow}^+$  is a nondecreasing sequence in  $\mathcal{S}_n^+$ , that

$$I_1^+(s_\nu) = \sum_{\mu=0}^{\nu} \|\{q_\mu\}\| = 0$$

and that

$$1_Q = \sup s \leq \sup s.$$

Thus

$$\mathbf{1}(1_Q) \leq I_{n,\uparrow}^n(s) = 0.$$

**Proposition 1.4.**  $|I_n(s)| \leq \mathbf{r}(s)$  whenever  $s \in \mathcal{S}_n$ .

*Proof.* Suppose  $s \in \mathcal{S}_n$ . Then  $|I_n(s)| \leq I_n(|s|) = \mathbf{r}(s)$ .  $\square$

**Theorem 1.1.** Suppose  $A \in \mathcal{M}_n$ ,  $B$  is a nondecreasing sequence in  $\mathcal{M}_n$  and  $A \subset \cup_{\nu=0}^{\infty} B_\nu$ . Then

$$\|A\| \leq \sup_{\nu} \|B_\nu\|.$$

*Proof.* We define the sequence  $C$  in  $\mathcal{M}_n$  by letting  $C_0 = B_0$  and for each  $\nu \in \mathbb{N}^+$  letting  $C_\nu = B_\nu \sim B_{\nu-1}$ . Then  $C$  is disjointed and  $B_\nu = \cup_{\mu=0}^{\nu} C_\mu$  for each  $\nu \in \mathbb{N}$ .

Suppose  $1 < \lambda < \infty$ . Choose a compact multirectangle  $K$  such that  $K \subset A$  and  $\|A\| \leq \lambda \|K\|$ . For each  $\nu \in \mathbb{N}$  choose an open multirectangle  $U_\nu$  such that  $C_\nu \subset U_\nu$  and  $\|U_\nu\| \leq \lambda \|C_\nu\|$ . Then  $K \subset \cup_{\nu=0}^{\infty} U_\nu$  so there is  $N \in \mathbb{N}$  such that  $K \subset \cup_{\mu=0}^N U_\mu$ . Thus

$$\lambda^{-1} \|A\| \leq \|K\| \leq \|\cup_{\mu=0}^N U_\mu\| \leq \lambda \sum_{\mu=0}^N \|U_\mu\| \leq \lambda \sum_{\mu=0}^N \|C_\mu\| = \lambda \|B_N\|.$$

Owing to the arbitrariness of  $\lambda$  the Lemma is proved.  $\square$

**Corollary 1.1.** Suppose  $A \in \mathcal{M}_n$ ,  $\mathcal{B}$  is a countable subfamily of  $\mathcal{M}_n$  and  $A \subset \cup \mathcal{B}$ . Then

$$\|A\| \leq \sum_{B \in \mathcal{B}} \|B\|.$$

*Proof.* Let  $B$  be an enumeration of  $\mathcal{B}$  and, for each  $\nu \in \mathbb{N}$ , let  $C_\nu = \cup_{\mu=0}^{\nu} B_\mu$ . Then  $C$  is a nondecreasing sequence in  $\mathcal{M}_n$  whose union contains  $A$  so that, by the preceding Theorem,

$$\|A\| \leq \sup_{\nu} \|C_\nu\| \leq \sup_{\nu} \sum_{\mu=0}^{\nu} \|B_\mu\| = \sum_{\nu=0}^{\infty} \|B_\nu\|.$$

$\square$

**Theorem 1.2.** We have

$$I_n^+(s) \leq \mathbf{1}(s) \quad \text{for any } s \in \mathcal{S}_n^+.$$

*Proof.* Suppose  $s \in \mathcal{S}_n^+$ ,  $t \in \mathcal{S}_{n,\uparrow}^+$  and  $s \leq \sup t$ . Let  $Y = \mathbf{rng} s \sim \{0\}$  and for each  $y \in Y$  let  $t_y \in \mathcal{S}_{n,\uparrow}^+$  be such that  $(t_y)_\nu = 1_{\{s=y\}} t_\nu$  for  $\nu \in \mathbb{N}$ .

Suppose  $0 < \sigma < 1$ . For each  $y \in Y$  we have  $\{s = y\} \subset \cup_{\nu=0}^{\infty} \{(t_y)_\nu > \sigma y\}$ ; it follows from Theorem 1.1 that

$$\sigma y \|\{s = y\}\| \leq \sup_{\nu} \sigma y \|\{(t_y)_\nu > \sigma y\}\|.$$

Now for any  $y \in Y$  and  $\nu \in \mathbb{N}$  we have  $\sigma y 1_{\{(t_y)_\nu > \sigma y\}} \leq (t_y)_\nu$  so that

$$\sigma y ||\{s = y\}|| = I_n^+ (\sigma y 1_{\{(t_y)_\nu > \sigma y\}}) \leq I_n^+ ((t_y)_\nu) \leq I_{n,\uparrow}^n(t_y).$$

It follows that

$$\sigma I_n^+(s) = \sum_{y \in Y} \sigma y ||\{s = y\}|| \leq \sum_{y \in Y} I_{n,\uparrow}^n(t_y) = I_n^+ \left( \sum_{y \in Y} t_y \right) \leq I_n^+(t).$$

Owing to the arbitrariness of  $\sigma$  the Theorem is proved.  $\square$

**Corollary 1.2.** We have

$$|I_n(s)| \leq \mathbf{l}(s) \quad \text{for any } s \in \mathcal{S}_n.$$

*Proof.* This follows from the preceding Theorem since  $|I_n(s)| \leq I_n^+(|s|)$  for any  $s \in \mathcal{S}_n$ .  $\square$

**Definition 1.3.** We let

$$\mathbf{Riem}_n$$

be the set of  $f \in \mathcal{F}_n$  such that for each  $\epsilon > 0$  there is  $s \in \mathcal{S}_n$  such that  $\mathbf{r}(f - s) < \epsilon$ . Thus  $\mathbf{Riem}_n$  is the closure with respect to  $\mathbf{r}$  of  $\mathcal{S}_n$ .

We say  $f \in \mathcal{F}_n$  is **Riemann integrable** if  $f \in \mathbf{Riem}_n$ .

We let

$$\mathbf{Leb}_n$$

be the set of  $f \in \mathcal{F}_n$  such that for each  $\epsilon > 0$  there is  $s \in \mathcal{S}_n$  such that  $\mathbf{l}(f - s) < \epsilon$ . Thus  $\mathbf{Leb}_n$  is the closure with respect to  $\mathbf{l}$  of  $\mathcal{S}_n$ .

We say  $f \in \mathcal{F}_n$  is **Lebesgue integrable** if  $f \in \mathbf{Leb}_n$ .

**Example 1.2.** Let  $Q$  be as in the preceding Example. It follows from the foregoing that

$$1_Q \in \mathbf{Leb}_1.$$

I claim that

$$1_Q \notin \mathbf{Riem}_1.$$

Suppose  $s \in \mathcal{S}_1$ ,  $m \in \mathcal{S}_1^+$  and  $|1_Q - s| \leq m$ . Suppose  $y \in \mathbb{R}$ ,  $z \in [0, \infty)$  and  $I = \mathbf{int} s^{-1}[\{y\}] \cap m^{-1}[\{z\}]$ . Then

$$|1 - y| = |1_Q(x) - s(x)| \leq m(x) = z \quad \text{if } x \in I \cap (0, 1) \cap \mathbb{Q}$$

and

$$|y| = |1_Q(x) - s(x)| \leq m(x) = z \quad \text{if } x \in I \cap (0, 1) \sim \mathbb{Q}$$

from which it follows that  $1/2 \leq z$  whenever  $x \in I$ . Thus  $1/2 \leq I_1^+(m)$ .

**Theorem 1.3.**  $\mathbf{Riem}_n$  is a linear subspace of  $\mathcal{F}_n$  and there is one and only one linear function

$$\mathbf{R} : \mathbf{Riem}_n \rightarrow \mathbb{R}$$

such that

- (i)  $\mathbf{R}(s) = I_n(s)$  whenever  $s \in \mathcal{S}_n$ ;
- (ii)  $|\mathbf{R}(f)| \leq \mathbf{r}(f)$  whenever  $f \in \mathbf{Riem}_n$ .

$\mathbf{Leb}_n$  is a linear subspace of  $\mathcal{F}_n$  and there is one and only one linear function

$$\mathbf{L} : \mathbf{Leb}_n \rightarrow \mathbb{R}$$

such that

- (i)  $\mathbf{L}(s) = I_n(s)$  whenever  $s \in \mathcal{S}_n$ ;
- (ii)  $|\mathbf{L}(f)| \leq \mathbf{l}(f)$  whenever  $f \in \mathbf{Leb}_n$ .

*Proof.* Two applications of the Abstract Closure Principle.  $\square$

**Remark 1.3.** Suppose  $f \in \mathbf{Riem}_n$  and  $\epsilon > 0$ . Choose  $s \in \mathcal{S}_n$  such that  $\mathbf{r}(f-s) \leq \epsilon$ . Then

$$|\mathbf{R}(f) - I_n(s)| = |\mathbf{R}(f) - \mathbf{R}(s)| = |\mathbf{R}(f-s)| \leq \mathbf{r}(f-s) \leq \epsilon.$$

Suppose  $f \in \mathbf{Leb}_n$  and  $\epsilon > 0$ . Choose  $s \in \mathcal{S}_n$  such that  $\mathbf{l}(f-s) \leq \epsilon$ . Then

$$|\mathbf{L}(f) - I_n(s)| = |\mathbf{L}(f) - \mathbf{L}(s)| = |\mathbf{L}(f-s)| \leq \mathbf{l}(f-s) \leq \epsilon.$$

**Definition 1.4.** Suppose  $a, b \in \mathbb{R}$ . We let

$$a \wedge b = \min\{a, b\} \quad \text{and we let} \quad a \vee b = \max\{a, b\}.$$

**Remark 1.4.** Note that

$$a \vee b + a \wedge b = a + b \quad \text{whenever } a, b \in \mathbb{R}.$$

**Proposition 1.5.** Suppose  $f \in \mathcal{F}_n$  and  $f \geq 0$ . Then

$$f \in \mathbf{Riem}_n \Leftrightarrow \text{for each } \epsilon > 0 \text{ there is } s \in \mathcal{S}_n^+ \text{ such that } \mathbf{r}(f-s) < \epsilon$$

and

$$f \in \mathbf{Leb}_n \Leftrightarrow \text{for each } \epsilon > 0 \text{ there is } s \in \mathcal{S}_n^+ \text{ such that } \mathbf{l}(f-s) < \epsilon.$$

and

*Proof.* Suppose  $f \in \mathbf{Riem}_n$  and  $\epsilon > 0$ . Choose  $t \in \mathcal{S}_n$  such that  $\mathbf{r}(f-t) < \epsilon$  and let  $s = t \wedge 0 \in \mathcal{S}_n^+$ . Since  $|f-s| \leq |f-t|$  we find that  $\mathbf{r}(f-s) \leq \mathbf{r}(f-t) < \epsilon$ .

If  $f \in \mathbf{Leb}_n$  the same argument works with  $\mathbf{r}$  replaced by  $\mathbf{l}$ .  $\square$

**Proposition 1.6.** Suppose  $f \in \mathcal{F}_n$  and  $f \geq 0$ . Then

$$f \in \mathbf{Riem}_n \Rightarrow \mathbf{R}(f) \geq 0 \quad \text{and} \quad f \in \mathbf{Leb}_n \Rightarrow \mathbf{L}(f) \geq 0.$$

*Proof.* Suppose  $f \in \mathbf{Riem}_n$  and  $\epsilon > 0$ . By the preceding Proposition there is  $s \in \mathcal{S}_n^+$  such that  $\mathbf{r}(f-s) < \epsilon$ . Thus

$$|\mathbf{R}(f) - \mathbf{R}(s)| = |\mathbf{R}(f-s)| \leq \mathbf{r}(f-s) < \epsilon$$

so that

$$\mathbf{R}(f) \geq \mathbf{R}(s) - \epsilon = I_n(s) - \epsilon \geq \epsilon.$$

Owing to the arbitrariness of  $\epsilon$  we infer that  $\mathbf{R}(f) \geq 0$ .

In the same way one shows that if  $f \in \mathbf{Leb}_n$  then  $\mathbf{L}(f) \geq 0$ .  $\square$

**Corollary 1.3.** We have

$$f, g \in \mathbf{Riem}_n \text{ and } f \leq g \Rightarrow \mathbf{R}(f) \leq \mathbf{R}(g)$$

and

$$f, g \in \mathbf{Leb}_n \text{ and } f \leq g \Rightarrow \mathbf{L}(f) \leq \mathbf{L}(g).$$

*Proof.* Apply the preceding Proposition to  $g-f$ .  $\square$

**Theorem 1.4.** Suppose  $f \in \mathbf{Riem}_n$ . Then  $f \in \mathbf{Leb}_n$  and

$$\mathbf{R}(f) = \mathbf{L}(f).$$

**Exercise 1.1.** Prove Theorem 1.4.

**Exercise 1.2.** Show that the product of two Riemann integrable functions is Riemann integrable.

**Theorem 1.5.** Suppose  $f \in \mathcal{F}_n$ ,  $f \geq 0$  and  $0 \leq c < \infty$ . Then

$$f \in \mathbf{Riem}_n \Rightarrow f \wedge c \in \mathbf{Riem}_n$$

and

$$f \in \mathbf{Leb}_n \Rightarrow f \wedge c \in \mathbf{Leb}_n.$$

Suppose  $f, g \in \mathcal{F}_n$ . Then

$$f, g \in \mathbf{Riem}_n \Rightarrow f \wedge g, f \vee g \in \mathbf{Riem}_n$$

and

$$f, g \in \mathbf{Leb}_n \Rightarrow f \wedge g, f \vee g \in \mathbf{Leb}_n.$$

**Lemma 1.1.** Suppose  $a, b, c, d \in \mathbb{R}$ . Then

$$\begin{aligned} |a \wedge b - a \wedge d| &\leq |b - d|, \\ |a \wedge b - c \wedge d| &\leq |a - c| + |b - d|, \\ |a \vee b - a \vee d| &\leq |b - d|, \\ |a \vee b - c \vee d| &\leq |a - c| + |b - d|. \end{aligned}$$

*Proof.* To prove the first inequality, suppose  $b < d$  and consider the three cases  $a \leq b$ ,  $a < b < d$ ,  $d \leq a$ ; then note that the inequality is symmetric in  $b$  and  $d$ .

To prove the second inequality note that

$$|a \wedge b - c \wedge d| \leq |a \wedge b - a \wedge d| + |a \wedge d - c \wedge d|$$

and then use the first inequality.

One may use the same techniques to prove the third and fourth inequality.  $\square$

**Exercise 1.3.** Prove Theorem 1.5. Make use of the preceding Remark and Lemma.

**Definition 1.5.** Suppose  $A \subset \mathbb{R}^n$  and  $f$  is a real valued function whose domain contains  $A$ . We let

$$f_A : \mathbb{R}^n \rightarrow \mathbb{R}$$

be such that

$$f_A(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{else.} \end{cases}$$

We set

$$\mathbf{R}_A(f) = \mathbf{R}(f_A) \quad \text{if } f_A \in \mathbf{Riem}_n$$

in which case we say  $f$  is **Riemann integrable over**  $A$  and we set

$$\mathbf{L}_A(f) = \mathbf{L}(f_A) \quad \text{if } f_A \in \mathbf{Leb}_n$$

in which case we say  $f$  is **Lebesgue integrable over**  $A$ . So, for example, if  $A = (a, b)$ ,

$$\int_a^b f(x) dx$$

is, by definition, one or both of  $\mathbf{R}_{(a,b)}(f)$  or  $\mathbf{L}_{(a,b)}(f)$ .

**Proposition 1.7.** Suppose  $f \in \mathcal{F}_n$  and  $S \in \mathcal{M}_n$ . Then

$$f \in \mathbf{Riem}_n \Rightarrow 1_S f \in \mathbf{Riem}_n$$

and

$$f \in \mathbf{Leb}_n \Rightarrow 1_S f \in \mathbf{Leb}_n.$$

**Exercise 1.4.** Prove Proposition 1.7.

1.1. **Outer measure. Sets of measure zero.** This will come in handy.

**Definition 1.6.** Suppose  $A \subset \mathbb{R}^n$ . We let

$$|A|^* = \inf \left\{ \sum_{\nu=0}^{\infty} \|R_\nu\| : R \text{ is a sequence in } \mathcal{R}_n \text{ and } A \subset \bigcup_{\nu=0}^{\infty} R_\nu \right\}$$

and call this nonnegative extended real number the **(Lebesgue) outer measure of  $A$** . We say  $A$  has **measure zero** if  $|A|^* = 0$ .

**Proposition 1.8.** Suppose  $A \subset \mathbb{R}^n$ . Then

$$|A|^* = \inf \left\{ \sup_{\nu} \|M_\nu\| : M \text{ is a nondecreasing sequence in } \mathcal{M}_n \text{ and } A \subset \bigcup_{\nu=0}^{\infty} M_\nu \right\}.$$

*Proof.* Let  $m$  be the right hand side of (1.8).

Suppose  $R$  is a sequence in  $\mathcal{R}_n$ . For each  $\nu \in \mathbb{N}$  let  $M_\nu = \bigcup_{\mu=0}^{\nu} R_\mu$ . Then  $M$  is a nondecreasing sequence in  $\mathcal{M}_n$ ,  $\bigcup_{\nu=0}^{\infty} R_\nu = \bigcup_{\nu=0}^{\infty} M_\nu$  and

$$\sup_{\nu} \|M_\nu\| \leq \sup_{\nu} \sum_{\mu=0}^{\nu} \|R_\mu\| = \sum_{\nu=0}^{\infty} \|R_\nu\|.$$

Suppose  $M$  is a nondecreasing sequence in  $\mathcal{M}_n$ . Let  $L_0 = M_0$  and for each  $\nu \in \mathbb{N}^+$  let  $L_\nu = M_\nu \sim M_{\nu-1}$ . Then  $L$  is a disjointed sequence in  $\mathcal{M}_n$  and  $\bigcup_{\nu=0}^{\infty} L_\nu = \text{cup}_{\nu=0}^{\infty} M_\nu$ . For each  $\nu \in \mathbb{N}$  choose  $N_\nu \in \mathbb{N}^+$  and disjointed rectangles  $S_{\nu,0}, \dots, S_{\nu, N_\nu-1}$  such that  $L_\nu = \bigcup_{\mu=1}^{N_\nu} S_{\nu,\mu}$ . Let  $R$  be the sequence in  $\mathcal{R}_n$  such that, for  $\nu \in \mathbb{N}$ ,  $R_{\sum_{\xi < \nu} + \mu} = S_{\nu,\mu}$  if  $0 \leq \mu < N_\nu$ . Then  $\bigcup_{\xi=0}^{\infty} R_\xi = \bigcup_{\nu=0}^{\infty} L_\nu$  and

$$\sum_{\nu=0}^{\infty} \|R_\nu\| = \sum_{\nu=0}^{\infty} \|L_\nu\| = \sup_{\nu} \sum_{\mu=0}^{\nu} \|L_\mu\| = \sup_{\nu} \|M_\nu\|.$$

The Proposition follows.  $\square$

**Theorem 1.6.** Suppose  $M$  is a multirectangle in  $\mathbb{R}^n$ . Then

$$|M|^* = \|M\|.$$

*Proof.* This follows directly from Corollary 1.1.  $\square$

**Theorem 1.7.** Suppose  $A \subset \mathbb{R}^n$ . Then  $|A|^* = \mathbf{I}^+(1_A)$ .

*Proof.* Suppose  $t \in \mathcal{S}_{n,\uparrow}^+$  and  $1_A \leq \sup t$ . Suppose  $0 < \sigma < \infty$ . For each  $\nu \in \mathbb{N}$ , let  $M_\nu = \{t_\nu > \sigma\} \in \mathcal{M}_n$ , note that  $\sigma 1_{M_\nu} \leq 1_{\{t_\nu > \sigma\}}$  so that  $\sigma \|M_\nu\| \leq I_n^+(t_\nu)$ . Now  $A \subset \bigcup_{\nu=0}^{\infty} M_\nu$  and  $M$  is an increasing sequence in  $\mathcal{M}_n$  so

$$\sigma |A|^* \leq \sigma \sup_{\nu} \|M_\nu\| = \sup_{\nu} I_\nu(t_\nu) = I_{n,\uparrow}^n(t).$$

Owing to the arbitrariness of  $\sigma$  it follows Proposition 1.8 that  $|A|^* \leq \mathbf{I}^+(1_A)$ .

On the other hand, suppose  $B$  is a sequence in  $\mathcal{R}_n$  such that  $A \subset \cup_{\nu=0}^{\infty} B_{\nu}$ . Let  $t$  be the sequence such that, for each  $\nu \in \mathbb{N}$ ,  $t_{\nu} = \sum_{\mu=0}^{\nu} 1_{B_{\mu}}$ . Evidently,  $t \in \mathcal{S}_{n,\uparrow}^+$  and  $1_A \leq \text{sup} t$ . Thus

$$\mathbf{I}^+(1_A) \leq I_{n,\uparrow}^+(t) = \sum_{\nu=0}^{\infty} I_n^+(1_{B_{\nu}}) = \sum_{\nu=0}^{\infty} \|B_{\nu}\|.$$

It follows that  $\mathbf{I}^+(1_A) \leq |A|^*$ .  $\square$

**Proposition 1.9.** The following statements hold.

- (i) If  $A \subset B \subset \mathbb{R}^n$  then  $|A|^* \leq |B|^*$ .
- (ii) If  $\mathcal{A}$  is a countable family of subsets of  $\mathbb{R}^n$  then

$$|\cup \mathcal{A}|^* \leq \sum_{A \in \mathcal{A}} |A|^*.$$

- (iii) The union of a countable family of sets of measure zero is a set of measure zero.
- (iv) Any countable set is a set of measure zero.
- (v) If  $a \in \mathbb{R}^n$  and  $A \subset \mathbb{R}^n$  then  $|a + A|^* = |A|^*$ . (That is, outer measure is **translation invariant.**)

*Proof.* (i) is immediate and (iii) and (iv) follow directly from (ii).

To prove (ii), suppose  $\mathcal{A}$  is countable family of subsets of  $\mathbb{R}^n$  and  $A$  is an enumeration of  $\mathcal{A}$ . Let  $\epsilon > 0$ . For each  $\nu \in \mathbb{N}$  choose a countable subfamily  $\mathcal{S}_{\nu}$  of  $\mathcal{R}_n$  such that

$$\sum_{S \in \mathcal{S}_{\nu}} \|S\| \leq |A_{\nu}|^* + \frac{\epsilon}{2^{n+1}}.$$

Then  $\mathcal{R} = \cup_{\nu=0}^{\infty} \mathcal{S}_{\nu}$  is a countable subfamily of  $\mathcal{R}_n$  whose union contains  $\cup \mathcal{A}$  and

$$\sum_{R \in \mathcal{R}} \|R\| \leq \sum_{\nu=0}^{\infty} \sum_{S \in \mathcal{S}_{\nu}} \|S\| \leq \sum_{\nu=0}^{\infty} \left( |A_{\nu}|^* + \frac{\epsilon}{2^{n+1}} \right) = \sum_{\nu=0}^{\infty} |A_{\nu}|^* + \epsilon.$$

Owing to the arbitrariness of  $\epsilon$  we see that (iii) holds.

(v) follows from the fact that  $\|a + R\|_n = \|R\|$  whenever  $a \in \mathbb{R}^n$  and  $R$  is a rectangle.  $\square$

**1.2. Nonmeasurable sets.** There exists a countable disjointed family  $\mathcal{C}$  of subsets of  $\mathbb{R}$  such that

- (i) there is  $c \in (0, \infty)$  such that  $|C|^* = c$  for  $C \in \mathcal{C}$ ;
- (ii)  $0 < |\cup \mathcal{C}|^* < \infty$ ;

it follows that

$$|\cup \mathcal{C}|^* < \sum_{C \in \mathcal{C}} |C|^* = \infty.$$

**Remark 1.5.** I'm fairly sure this is equivalent to certain forms of the Axiom of Choice. Consult Professor Hodel, the local set theory expert, if you want more information about this.

*Proof.* We're going to be terse! Let  $B$  be the range of a choice function for

$$\{x + \mathbb{Q} : x \in \mathbb{R}\}.$$

(In the parlance of algebra,  $B$  is a set of coset representatives for the reals modulo the rationals.) It follows that

$$\{q + B : q \in \mathbb{Q}\}$$

is a countable partition of  $\mathbb{R}$ . Now

$$\infty = |\mathbb{R}|^* \leq \sum_{q \in \mathbb{Q}} |q + B|^* = \sum_{q \in \mathbb{Q}} |B|^*$$

so

$$|B|^* > 0.$$

Since

$$|B|^* \leq \sum_{z \in \mathbf{Z}} |B \cap [z, z + 1)]|^*$$

there is  $z \in \mathbf{Z}$  such that

$$|B \cap [z, z + 1)]|^* > 0.$$

For each  $q \in \mathbb{Q}$  let

$$C_q = q + (B \cap [z, z + 1)).$$

Then

$$|C_q|^* = |C_0|^* > 0 \quad \text{whenever } q \in \mathbb{Q}.$$

Let

$$\mathcal{C} = \{C_q : q \in \mathbb{Q}, 0 \leq q \leq 1\}.$$

Then

$$|\cup \mathcal{C}|^* \leq |[z, z + 2)]|^* = 2$$

but

$$\sum_{C \in \mathcal{C}} |C|^* = \infty |C_0|^* = \infty.$$

□

**1.3. More on the Riemann integral.** The Riemann integral isn't so great but everybody studies it because it's easier to define.

The Definition we gave of the Riemann integral is not the standard one. Now we show that it is equivalent to the standard one.

**Definition 1.7.** Suppose  $f \in \mathcal{B}_n$  and  $\delta > 0$ . We let

$$\mathbf{RiemSum}_\delta(f)$$

be the set of sums

$$\sum_{R \in \mathcal{R}} f(c(R)) |R|$$

where

- (i)  $\mathcal{R}$  is a finite nonoverlapping family of nonempty bounded rectangles;
- (ii)  $\mathbf{diam} R < \delta$  whenever  $R \in \mathcal{R}$ ;
- (iii)  $\{f \neq 0\} \subset \cup \mathcal{R}$ ;
- (iv)  $c$  is a choice function for  $\mathcal{R}$ .

The members of  $\mathbf{RiemSum}_\delta(f)$  are called **Riemann sums for  $f$  with mesh diameter at most  $\delta$** .

**Theorem 1.8.** Suppose  $f \in \mathcal{B}_n$ . Then  $f \in \mathbf{Riem}_n$  if and only if

$$\inf\{\mathbf{diam\ RiemSum}_\delta(f) : \delta > 0\} = 0$$

in which case

$$\bigcap_{0 < \delta < \infty} \mathbf{RiemSum}_\delta(f) = \{R(f)\}.$$

*Proof. Part One.* Suppose  $\inf\{\mathbf{diam\ RiemSum}_\delta(f) : 0 < \delta < \infty\} = 0$ . Let  $\delta > 0$  and let  $\mathcal{R}$  be a finite disjointed family of nonempty rectangles such that  $\{f \neq 0\} = \cup \mathcal{R}$  and such that  $\mathbf{diam}\ R < \delta$  for  $R \in \mathcal{R}$ . Suppose  $\eta \in (0, \infty)$ .

Let  $\underline{c}$  and  $\bar{c}$  be choice functions for  $\mathcal{R}$  such that

$$f(\underline{c}(R)) \leq \inf_R f + \eta \quad \text{and} \quad \sup_R f \leq f(\bar{c}(R)) + \eta \quad \text{whenever } R \in \mathcal{R}.$$

Let

$$\underline{S} = \sum_{R \in \mathcal{R}} f(\underline{c}(R))|R|; \quad \text{let} \quad \bar{S} = \sum_{R \in \mathcal{R}} f(\bar{c}(R))|R|;$$

let

$$s = \sum_{R \in \mathcal{R}} (\inf_R f)1_R \in \mathcal{S}_n; \quad \text{and let} \quad m = \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f)1_R \in \mathcal{S}_n^+.$$

Then  $|f - s| \leq m$  and, since  $\underline{S}, \bar{S} \in \mathbf{RiemSum}_\delta(f)$ , we find that

$$\begin{aligned} I(m) &= \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f)|R| \\ &\leq \sum_{R \in \mathcal{R}} (f(\bar{c}(R)) - f(\underline{c}(R)) + 2\eta)|R| \\ &\leq \bar{S} - \underline{S} + 2\eta|T| \\ &\leq \mathbf{diam\ RiemSum}_\delta(f) + 2\eta|T|. \end{aligned}$$

Owing to the arbitrariness of  $\delta$  and  $\eta$ , it follows that  $f \in \mathbf{Riem}_n$ .

**Part Two.** Suppose  $f \in \mathbf{Riem}_n$  and  $\epsilon > 0$ . Choose  $s \in \mathcal{S}_n$ ,  $m \in \mathcal{S}_n^+$  such that  $|f - s| \leq m$ ,  $I_n^+(m) < \epsilon/4$ . We will show that there is  $\delta > 0$  such that if  $\Sigma \in \mathbf{Riemsum}_\delta(f)$  then  $|\Sigma - I(s)| < \epsilon/2$ ; that will imply that  $\mathbf{diam\ Riemsum}_\delta(f) < \epsilon$ .

Let  $B = \max \mathbf{rng}(|s| + m)$  and note that  $|f| \leq |s| + m \leq B$ . Let  $\eta > 0$  be such that  $2B\eta < \epsilon/4$  and let  $N = \{s \neq 0\} \cup \{m > 0\}$ .

Let  $Z$  be a bounded rectangle such that  $\mathbf{cl}\ N \subset \mathbf{int}\ Z$ . Choose choose  $\mathcal{Q}, \sigma, \mu$  such that

- (i)  $\mathcal{Q}$  is a finite disjointed family of rectangles and  $\cup \mathcal{Q} = Z$ ;
- (ii)  $\sigma : \mathcal{Q} \rightarrow \mathbb{R}$  and  $s = \sum_{Q \in \mathcal{Q}} \sigma(Q)1_Q$ ;
- (iii)  $\mu : \mathcal{Q} \rightarrow [0, \infty)$  and  $m = \sum_{Q \in \mathcal{Q}} \mu(Q)1_Q$ .

For each  $Q \in \mathcal{Q}$  choose an rectangle  $Q_{\text{in}}$  such that  $\mathbf{cl}\ Q_{\text{in}} \subset \mathbf{int}\ Q$  and such that if  $Y = \cup\{Q_{\text{in}} : Q \in \mathcal{Q}\}$  then

$$\|Z \sim Y\| < \eta.$$

Let  $\delta > 0$  be such that

$$X \subset \mathbb{R}^2, \quad \mathbf{diam}\ X < \delta, \quad \text{and} \quad X \cap N \neq \emptyset \Rightarrow X \subset Z$$

and such that

$$X \subset \mathbb{R}^2, \quad \mathbf{diam}\ X < \delta, \quad \text{and} \quad X \cap Y \neq \emptyset \Rightarrow X \subset Q \quad \text{for some } Q \in \mathcal{Q}.$$

Suppose  $\mathcal{R}, \gamma$  are as in Definition 1.7. Let

$$\Sigma = \sum_{R \in \mathcal{R}} f(c(R)) \|R\|.$$

We may assume without loss of generality that

$$\cup \mathcal{R} = Z.$$

Indeed, if  $R \in \mathcal{R}$  and  $R$  meets the complement of  $Z$  then  $R \cap N = \emptyset$  so  $f(c(R)) = 0$  so we may delete  $R$  from  $\mathcal{R}$  without changing  $\Sigma$ . Moreover, since  $f$  vanishes on  $Z \sim \cup \mathcal{R}$ , if we adjoin to  $\mathcal{R}$  a finite disjointed family of nonempty rectangles  $\mathcal{S}$  with union  $Z \sim \cup \mathcal{R}$  and if we extend  $c$  to  $\mathcal{R} \cup \mathcal{S}$  then

$$\sum_{S \in \mathcal{R} \cup \mathcal{S}} f(d(S)) = \sum_{R \in \mathcal{R}} f(c(S)) = \sigma$$

since  $f(d(S)) = 0$  for  $S \in \mathcal{S}$ .

Let

$$\mathcal{R}_{\text{in}} = \{R \in \mathcal{R} : R \subset Q \text{ for some } Q \in \mathcal{Q}\}$$

and let  $\mathcal{R}_{\text{out}} = \mathcal{R} \sim \mathcal{R}_{\text{in}}$ . Let  $q : \mathcal{R}_{\text{in}} \rightarrow \mathcal{Q}$  be such that  $R \subset q(R)$  for  $R \in \mathcal{R}_{\text{in}}$ ;  $q$  is well defined since  $\mathcal{Q}$  is disjointed. We have

$$\begin{aligned} I_n(s) &= \sum_{Q \in \mathcal{Q}} \sigma(Q) \|Q\| \\ (1) \quad &= \sum_{Q \in \mathcal{Q}} \sigma(Q) \sum_{R \in \mathcal{R}} \|Q \cap R\| \\ &= \sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{in}}} \sigma(Q) \|Q \cap R\| + \sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{out}}} \sigma(Q) \|Q \cap R\| \end{aligned}$$

as well as

$$\begin{aligned} \Sigma &= \sum_{R \in \mathcal{R}} f(c(R)) \|R\| \\ (2) \quad &= \sum_{R \in \mathcal{R}} f(c(R)) \sum_{Q \in \mathcal{Q}} \|Q \cap R\| \\ &= \sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{in}}} f(c(R)) \|Q \cap R\| + \sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{out}}} f(c(R)) \|Q \cap R\|. \end{aligned}$$

Now

$$\begin{aligned} (3) \quad &\left| \sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{in}}} \sigma(Q) - f(c(R)) \|Q \cap R\| \right| \\ &\leq \sum_{R \in \mathcal{R}_{\text{in}}} |\sigma(q(R)) - f(c(R))| \|q(R)\| \\ &\leq \sum_{R \in \mathcal{R}_{\text{in}}} |\mu(q(R))| \|q(R)\| \\ &\leq I_n^+(m) \\ &< \frac{\epsilon}{4}. \end{aligned}$$

Moreover,

$$\sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{out}}} |\sigma(Q)| \|Q \cap R\| \leq B \|Z \sim Y\| < B\eta$$

and that

$$\sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}_{\text{out}}} |f(c(R))| \|Q \cap R\| \leq B \|Z \sim Y\| < B\eta.$$

Thus

$$|\Sigma - I(s)| < \frac{\epsilon}{4} + 2B\eta < \frac{\epsilon}{2}.$$

□

#### 1.4. The fundamental theorems of calculus.

**Theorem 1.9.** Suppose  $-\infty < a < b < \infty$ ,  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f$  is differentiable at each point of  $(a, b)$  and  $f'$  is Riemann integrable on  $(a, b)$ . Then

$$(4) \quad \mathbf{R}_{(a,b)}(f') dx = f(b) - f(a).$$

**Remark 1.6.** Using more traditional notation, (4) says

$$\int_a^b f'(x) dx = f(b) - f(a).$$

**Remark 1.7.** Suppose  $-\infty < a < b < \infty$ ,  $f : (a, b) \rightarrow \mathbb{R}$ ,  $f$  is differentiable at each point of  $(a, b)$  and  $f'$  is Riemann integrable on  $(a, b)$ . Then there is  $M \in [0, \infty)$  such that  $|f'(x)| \leq M$  whenever  $a < x < b$ . This implies  $|f(x) - f(y)| \leq M|x - y|$  whenever  $a < x < y < b$  which is to say that **Lip**  $f \leq M$ . In particular,  $f$  has a unique continuous extension to the closure  $[a, b]$  of  $(a, b)$ .

**Exercise 1.5.** Prove Theorem 1.9. Note that

$$f(b) - f(a) = \sum_{i=1}^N f(x_i) - f(x_{i-1})$$

whenever  $N \in \mathbb{N}^+$  and  $a = x_0 \leq x_1 \leq \dots \leq x_N = b$ . Use the Mean Value Theorem to construct Riemann sums which do the job.

**Theorem 1.10.** Suppose  $f : (a, b) \rightarrow \mathbb{R}$ ,  $f$  is Riemann integrable and

$$F(x) = \mathbf{R}_{(a,x)}(f) \quad \text{for } x \in (a, b).$$

Then

$$(5) \quad F'(x) = f(x) \quad \text{whenever } x \in (a, b) \text{ and } f \text{ is continuous at } x.$$

**Remark 1.8.** Using more traditional notation, (4) says

$$\frac{d}{dx} \left( \int_a^x f'(t) dt \right) = f(x).$$

**Exercise 1.6.** Prove Theorem 1.10. Don't hesitate to use the theory already developed.

**Exercise 1.7.** Suppose  $1 < p < \infty$ . Let  $f, g \in \mathcal{F}_n^+$  be such that

$$f(x) = \begin{cases} \frac{1}{x^p} & \text{if } 0 < x < 1, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{x^p} & \text{if } 1 < x < \infty, \\ 0 & \text{else.} \end{cases}$$

Show that  $\mathbf{I}^+(f) < \infty$  if and only if  $p < 1$  and show that  $\mathbf{I}^+(g) < \infty$  if and only if  $p > 1$ .

(Big Hint: Use Theorem 1.9 together with the Monotone Convergence Theorem of the next set of notes.)

**1.5. Characterization of Riemann integrability.** The following Theorem characterizes  $\mathbf{Riem}_n$  in a very precise way.

**Theorem 1.11.** Suppose  $f \in \mathcal{B}_n$ . Then  $f \in \mathbf{Riem}_n$  if and only if the set of discontinuities of  $f$  has measure zero.

We will now prove this Theorem. So suppose  $f \in \mathcal{B}_n$ . Let  $M \in [0, \infty)$  be such that  $|f| \leq M$  and let  $S$  be a compact rectangle such that  $\{f \neq 0\} \subset S$ . For each positive integer  $\nu$  let

$$D_\nu = \{x \in \mathbb{R}^n : \mathbf{osc}f(x) \geq 1/\nu\}$$

and let  $E = \bigcup_{\nu=1}^{\infty} D_\nu$ . Then  $E$  is the set of discontinuities of  $f$ .

Suppose  $\nu \in \mathbb{N}^+$ . By an earlier exercise about  $\mathbf{osc}f$ ,  $D_\nu$  is closed. Since  $D_\nu \subset \{f \neq 0\}$  we find that  $D_\nu$  is bounded. Thus  $D_\nu$  is compact.

**Lemma 1.2.** Suppose  $f \in \mathbf{Riem}_n$ . There is a disjointed family  $\mathcal{R}$  of rectangles such that  $\bigcup \mathcal{R} = S$  and

$$\sum_{R \in \mathcal{R}} (\sup f[R] - \inf f[R])|R| \leq \epsilon.$$

*Proof.* Let  $s \in \mathcal{S}_n$  and  $m \in \mathcal{S}_n^+$  be such that  $|f - s| \leq m$  and  $I_n^+(m) \leq \epsilon/2$ . Replacing  $s$  and  $m$  by  $1_S s$  and  $1_S m$  if necessary we may assume without loss of generality that  $\{s \neq 0\} \cup \{m > 0\} \subset S$ . Choose  $\mathcal{R}, \sigma, \mu$  such that  $\mathcal{R}$  is a finite disjointed family of rectangles with union  $S$ ;  $\sigma, \mu$  are functions with domain  $\mathcal{R}$  and ranges contained in  $\mathbb{R}$  and  $[0, \infty)$ , respectively;

$$s = \sum_{R \in \mathcal{R}} \sigma(R)1_R \quad \text{and} \quad m = \sum_{R \in \mathcal{R}} \mu(R)1_R.$$

Suppose  $R \in \mathcal{R}$ . Then

$$\sigma(R) - \mu(R) = s(x) - m(x) \leq f(x) \leq s(x) + m(x) = \sigma(R) + \mu(R) \quad \text{for } x \in R.$$

This implies

$$\sigma(R) - \mu(R) \leq \inf_R f \quad \text{and} \quad \sup_R f \leq \sigma(R) + \mu(R)$$

so

$$(\sup_R f - \inf_R f)|R| \leq 2\mu(R)|R|.$$

Now sum over  $\mathcal{R}$ . □

**Corollary 1.4.** Suppose  $f \in \mathbf{Riem}_n$ . Then the set of discontinuities of  $f$  has measure zero.

*Proof.* Since  $|E|^* = |\bigcup_{\nu=1}^{\infty} D_\nu|^* \leq \sum_{\nu=1}^{\infty} |D_\nu|^*$  it will suffice to show that  $|D_\nu|^* = 0$  for all  $\nu \in \mathbb{N}^+$ .

So suppose  $\nu \in \mathbb{N}^+$  and let  $\epsilon > 0$ . Let  $\mathcal{R}$  be as in the preceding Theorem with  $\epsilon$  there replaced by  $\epsilon/\nu$ . I claim that

$$(6) \quad \frac{1}{\nu} |D_\nu \cap R|^* \leq (\sup_R f - \inf_R f)|R| \quad \text{whenever } R \in \mathcal{R}.$$

Suppose  $R \in \mathcal{R}$ . If  $x \in D_\nu \cap \mathbf{int} R \neq \emptyset$  we have  $1/\nu \leq \mathbf{osc}f(x) \leq \sup_R f - \inf_R f$ ; moreover,  $|D_\nu \cap R|^* \leq |R|^* = |R|$ . If  $D_\nu \cap \mathbf{int} R$  is empty then  $|D_\nu \cap R|^* \leq |\mathbf{bdry} R|^* = |\mathbf{bdry} R| = 0$ . Thus (6) holds. It follows that

$$|D_\nu|^* \leq \sum_{R \in \mathcal{R}} |D_\nu \cap R|^* \leq \nu \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f)|R| < \epsilon.$$

Owing to the arbitrariness of  $\epsilon$  we conclude that  $|D_\nu|^* = 0$ .  $\square$

**Lemma 1.3.** Suppose  $f \in \mathcal{B}_n$  and the set of discontinuities of  $f$  has measure zero. Then  $f \in \mathbf{Riem}_n$ .

*Proof.* Suppose  $\epsilon > 0$ . Choose  $\eta > 0$  and  $\nu \in \mathbb{N}^+$  such that  $\|S\|/\nu + M\eta < \epsilon$ . Let  $\mathcal{Z}$  be a countable family of open rectangles such that  $D_\nu \subset \cup \mathcal{Z}$  and  $\sum_{R \in \mathcal{Z}} \|R\| < \eta$ . Since  $D_\nu$  is compact there is a finite subfamily  $\mathcal{F}$  of  $\mathcal{Z}$  such that  $D_\nu \subset \cup \mathcal{F}$ . Let  $\delta > 0$  be such that  $\delta$  is less than the Lebesgue number of the covering

$$\left\{ U : \text{is an open subset of } \mathbb{R}^n \text{ and } (\sup_U f - \inf_U f) < \frac{1}{\nu} \right\}.$$

of the compact set  $K = S \sim \cup \mathcal{F}$ . Let  $\mathcal{R}$  be a finite disjointed family of rectangles with union  $K$  none of whose diameters exceed  $\delta$ ; let

$$s = \sum_{S \in \mathcal{R}} \inf_R f 1_R \quad \text{and let} \quad m = \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f) 1_R + M \sum_{R \in \mathcal{F}} 1_R.$$

Then

$$|f - s| \leq m \quad \text{and} \quad I_n^+(m) \leq \frac{1}{\nu} \|S\| + M\eta < \epsilon.$$

$\square$

**Definition 1.8.** We say a subset  $A$  of  $\mathbb{R}^n$  has **Jordan content** if  $1_A \in \mathbf{Riem}_n$  in which case we let  $\mathbf{R}(1_A)$  be the **Jordan content of  $A$** . In view of the preceding Theorem,  $A$  will have Jordan content if and only if  $A$  is bounded and its boundary has measure zero. Since the boundary of such a set is compact we find that  $A$  has Jordan content if and only if  $A$  is bounded and for every  $\epsilon > 0$  there is a finite family  $\mathcal{R}$  of open rectangles such that  $\sum_{R \in \mathcal{R}} \|R\| \leq \epsilon$ . Since  $\mathbf{R}$  is linear Jordan content is additive and if  $A, B$  have Jordan content then so do  $A \cup B$ ,  $A \cap B$  and  $A \sim B$ .