Alternating and symmetric multilinear functions.

Suppose V and W are normed vector spaces.

Definition. For each integer p we set

$$\bigotimes^p (V;W) = \begin{cases} \{0\} & \text{if } p<0; \\ W & \text{if } p=0; \\ \mathbf{L}(\underbrace{V \ldots \cdots \ldots V}_{p \text{ times}};W) & \text{if } p \ge 1. \end{cases}$$

We say $\mu \in \bigotimes^p (V;W)$ is **symmetric** if either $p \le 1$ or $p > 1$ and the value of μ does not change if any pair of its arguments are transposed, and we say μ is **antisymmetric** or **alternating** if either $p \le 1$ or the value of μ is multiplied by -1 if any pair its arguments are transposed. We let

$$\bigodot^p(V;W) \quad \text{and} \quad \bigwedge^p(V;W)$$

be the sets of symmetric and alternating members of $\bigotimes^p V;W$, respectively. Evidently, $\bigodot^p(V;W)$ and $\bigwedge^p(V;W)$ are linear subspaces of $\bigotimes^p(V;W)$.

If U is a vector spaces and $l \in \mathbf{L}(U,V)$ we define the linear map

$$\bigotimes^p (l;W) : \bigotimes^p (V;W) \to \bigotimes^p (U;W)$$

by setting

$$\bigotimes^p (l;W)(\varphi)(u_1,\ldots,u_p) = \varphi(l(u_1),\ldots,l(u_p)), \quad \varphi \in \bigotimes^p (V;W),\ u_1,\ldots,u_p \in U.$$

This extends the notion of adjoint encountered previously. We note that $\bigotimes^p (l;W)$ preserves symmetry and antisymmetry and we set

$$\bigodot^p (l;W) = \bigotimes^p (l;W)\Big| \bigodot^p (V;W) \quad \text{and} \quad \bigwedge^p (l;W) = \bigotimes^p (l;W)\Big| \bigwedge^p (V;W).$$

One easily verifies that if Z is a vector space and $m \in \mathbf{L}(V;Z)$ then

$$\bigotimes^p (m \circ l;W) = \bigotimes^p (l;W) \circ \bigotimes^p (m;W)$$

and that similar formulae hold with $\bigotimes^p (\cdot;W)$ replaced by $\bigodot^p (\cdot;W)$ and $\bigwedge^p (\cdot;W)$.

Interior multiplication. For each integer p we define the bilinear map

$$\bigotimes^p(V;W) \times V \xrightarrow{\llcorner} \bigotimes^{p-1}(V;W)$$

as follows: Given $\varphi \in \bigotimes^p(V;W)$ and $v \in V$ we set $\varphi\llcorner = 0$ in case $p \le 0$, we set $\varphi\llcorner v = \varphi(v)$ in case $p = 1$ and, in case $p > 1$, we set

$$\varphi\llcorner v(v_2,\ldots,v_p) = \varphi(v,v_2,\ldots,v_p), \quad v_2,\ldots,v_p \in V.$$

We call $\varphi\llcorner v$ **interior multiplication** or **contraction of** φ **by** v. by requiring that

Note that interior multiplication by v preserves the subspaces of symmetric and alternating functions.

Proposition. Suppose $\mathcal{B}$ is a basis for V and $\varphi, \psi \in \bigotimes^p(V;W)$. Then $\varphi = \psi$ if and only if

$$\varphi(v(1),\ldots,v(p)) = \psi(v(1),\ldots,v(p))$$

for any map $v : \{1, \ldots, p\} \to \mathcal{B}$.
Proof. We hope this intuitively obvious to the reader. A short formal proof my proceed by induction on p using the fact that if $p \geq 1$ and $\varphi \llcorner v = \psi \llcorner v$ for all $v \in V$ then $\varphi = \psi$. □

Proposition. Suppose $\mathcal{B}$ is a basis for V and $\varphi, \psi \in \bigwedge^p(V; W)$. Then $\varphi = \psi$ if and only if for each p element subset B of $\mathcal{B}$ there is a univalent mapping v from $\{1, \ldots, p\}$ onto B such that

$$\varphi(v(1), \ldots, v(p)) = \psi(v(1), \ldots, v(p)).$$

Proof. We hope this intuitively obvious to the reader. A short formal proof could proceed by induction on p using the fact that if $p \geq 1$ and $\varphi \llcorner v = \psi \llcorner v$ for all $v \in V$ then $\varphi = \psi$. □

Definition. For each integer p we let

$$\bigotimes^p V = \bigotimes^p (V; \mathbf{R}), \quad \bigodot^p V = \bigodot^p (V; \mathbf{R}), \quad \bigwedge^p V = \bigwedge^p (V; \mathbf{R})$$

and we let

$$\bigotimes^p l = \bigotimes^p (l; \mathbf{R}), \quad \bigodot^p l = \bigodot^p (l; \mathbf{R}), \quad \bigwedge^p l = \bigwedge^p (l; \mathbf{R}).$$

Exterior algebra.

Let V be a vector space, let $n = \mathbf{dim}\, V$ and suppose $n < \infty$.

Proposition. Suppose $p \geq 2$, $v_i \in V$, $i = 1, \ldots, p$ and

$$\mathbf{dim}\,\mathbf{span}\{v_i : i \in \{1, \ldots, p\}\} < p.$$

Then

$$\varphi(v_1, \ldots, v_p) = 0 \quad \text{for any } \varphi \in \bigwedge^p V.$$

Proof. For some $j \in \{1, \ldots, p\}$ and for some scalars c_i, $i \in I$, with $I = \{1, \ldots, p\} \sim \{j\}$ we have $v_j = \sum_{i \in I} c_i\, v_i$. For each $i \in I$ we define $w_1^{(i)}, \ldots, w_p^{(i)}$ by setting

$$w_k^{(i)} = \begin{cases} v_i & \text{if } k = j, \\ v_k & \text{if } k \in I, \end{cases}, \ i \in I.$$

For each $i \in I$ we have $w_j^{(i)} = w_i^{(i)}$ so $\varphi(w^{(i)}) = 0$. Thus

$$\varphi(v_1, \ldots, v_p) = \sum_{i \in I} c_i \varphi(w_1^{(i)}, \ldots, w_p^{(i)}) = 0.$$

□

Corollary.

$$\bigwedge^p V = \{0\} \quad \text{if } p > n.$$

Theorem. The Contravariant Exterior Product. There is one and only one map

$$\bigwedge^p V \times \bigwedge^q V \xrightarrow{\wedge} \bigwedge^{p+q} V$$

such that, if $\varphi \in \bigwedge^p V$ and $\psi \in \bigwedge^q V$ then

(1) $\varphi \wedge \psi = \varphi \psi$ if $p = 0$ and $q = 0$;

(2) $(\varphi \wedge \psi) \llcorner v = (\varphi \llcorner v) \wedge \psi + (-1)^p \varphi \wedge (\psi \llcorner v)$ for all v in V.
This mapping is bilinear.

Remark. Because (2) holds we say $\varphi \mapsto \varphi \llcorner v$ is a **skewderivation**.
Proof. The statement holds trivially if $p < 0$ or $q < 0$ so suppose $p \geq 0$ and $q \geq 0$ and induct on $r = p + q$. It is evident by induction on r that there is unique map

$$\bigwedge^p V \times \bigwedge^q V \xrightarrow{\wedge} \bigotimes^{p+q} V$$

such that (1) and (2) are satisfied and that this map is bilinear. We need to show that if $\varphi \in \bigwedge^p V$ and $\psi \in \bigwedge^q V$ then $\varphi \wedge \psi$ is alternating. This is trivially the case if $r = 0$ so assume $r > 0$ and that the Theorem holds for smaller r.

Since

$$(\varphi \wedge \psi) \llcorner v = (\varphi \llcorner v) \wedge \psi + (-1)^p \varphi \wedge (\psi \llcorner v)$$

for any $v \in V$ the inductive hypothesis implies that $\varphi \wedge \psi$ is alternating in its last $r - 1$ slots. To complete the proof it will suffice to show that it alternating in its first two slots. That is, given $v, w \in V$ we need to show that

$$((\varphi \wedge \psi) \llcorner v) \llcorner w$$

is alternating in v, w. But

$$\begin{aligned}
&((\varphi \wedge \psi) \llcorner v) \llcorner w \\
&\qquad = \big((\varphi \llcorner v) \wedge \psi + (-1)^p \varphi \wedge (\psi \llcorner v)\big) \llcorner w \\
&\qquad = \big((\varphi \llcorner v) \llcorner w\big) \wedge \psi + (-1)^{p-1} (\varphi \llcorner v) \wedge (\psi \llcorner w) \\
&\qquad + (-1)^p (\varphi \llcorner w) \wedge (\psi \llcorner v) + (-1)^p (-1)^p \varphi \wedge \big((\psi \llcorner v) \llcorner w\big).
\end{aligned}$$

The sum of the second and third terms in this sum is clear alternating and v and w and the first and fourth terms in the summand are alternating by the inductive hypothesis. □

Theorem. Suppose $\varphi \in \bigwedge^p V$, $\psi \in \bigwedge^q V$ and $\zeta \in \bigwedge^r V$. Then

$$(\varphi \wedge \psi) \wedge \zeta = \varphi \wedge (\psi \wedge \zeta).$$

(That is, exterior multiplication is **associative**.)

Proof. The Theorem holds trivially if any of p, q, r are negative. So we assume that p, q, r are nonnegative and induct on $s = p + q + r$. The Theorem holds trivially if $s = 0$ so suppose $s > 0$ and that Theorem holds for smaller s. Given $v \in V$ we calculate

$$\begin{aligned}
\big((\varphi \wedge \psi) \wedge \zeta\big) \llcorner v &= \big((\varphi \wedge \psi) \llcorner v\big) \wedge \zeta + (-1)^{p+q} (\varphi \wedge \psi) \wedge (\zeta \llcorner v) \\
&= \big((\varphi \llcorner v) \wedge \psi\big) \wedge \zeta + (-1)^p \big(\varphi \wedge (\psi \llcorner v)\big) \wedge \zeta + (-1)^{p+q} (\varphi \wedge \psi) \wedge (\zeta \llcorner v);
\end{aligned}$$

$$\begin{aligned}
\big(\varphi \wedge (\psi \wedge \zeta)\big) \llcorner v &= (\varphi \llcorner v) \wedge (\psi \wedge \zeta) + (-1)^p \varphi \wedge \big((\psi \wedge \zeta) \llcorner v\big) \\
&= (\varphi \llcorner v) \wedge (\psi \wedge \zeta) + (-1)^p \varphi \wedge \big((\psi \llcorner v) \wedge \zeta\big) + (-1)^p (-1)^q \varphi \wedge \big(\psi \wedge (\zeta \llcorner v)\big).
\end{aligned}$$

Now apply the inductive hypothesis. □

Theorem. Suppose $\varphi \in \bigwedge^p V$ and $\psi \in \bigwedge^q V$. Then

$$\varphi \wedge \psi = (-1)^{pq} \psi \wedge \varphi.$$

(That is, exterior multiplication is **commutative** in the graded sense.)

Proof. The Theorem holds trivially if either p or 1 is negative. Induct on $r = p + q$. If $r = 0$ this amounts to the commutative law for multiplication of real numbers so suppose $r > 0$ and that the Theorem holds for smaller r. For any v in V we have

$$(\varphi \wedge \psi) \llcorner v = (\varphi \llcorner v) \wedge \psi + (-1)^p \varphi \wedge (\psi \llcorner v);$$

$$(-1)^{pq} (\psi \wedge \varphi) \llcorner v = (-1)^{pq} (\psi \llcorner v) \wedge \varphi + (-1)^{pq} (-1)^q \psi \wedge (\varphi \llcorner v).$$

Now apply the inductive hypothesis. $\square$

Corollary. Suppose p is odd and $\varphi \in \bigwedge^p V$. Then

$$\varphi \wedge \varphi = 0.$$

Remark. Let $V = \mathbf{R}^4$. We have

$$(\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4) \wedge (\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4) = 2\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4.$$

As we shall soon see, $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \neq 0$.

Definition. For each $p \in \{0, 1, \ldots, n\}$ we let

$$\Lambda(p, n)$$

equal the empty set if $p = 0$ and we let it equal the set of increasing maps from $\{1, \ldots, p\}$ into $\{1, \ldots, n\}$ if $p > 0$. Let

$$\Lambda(n) = \bigcup_{p=0}^{n} \Lambda(p, n).$$

Evidently,

$$\mathbf{card}\, \Lambda(p, n) = \binom{n}{p} \quad \text{whenever } p \in \{0, \ldots, p\}.$$

For each integer $p \geq 1$, $i \in \{1, \ldots, p\}$ and $\lambda \in \Lambda(p, n)$ we let λ_i be that member of $\Lambda(p-1, n)$ such that $\mathbf{rng}\, \lambda_i = \mathbf{rng}\, \lambda \sim \{\lambda(i)\}$.

Bases and Dimension.

Let V be a vector space, let $n = \mathbf{dim}\, V$ and let

$$v_1, \ldots, v_n \quad \text{and} \quad v^1, \ldots, v^n$$

be dual basics sequences for V and V^*, respectively.

For each p in $\{1, \ldots, n\}$ and for each λ in $\Lambda(p, n)$ let

$$v_{[\lambda]} \in \left(\bigwedge^p V\right)^*$$

be such that

$$v_{[\lambda]}(\varphi) = \varphi(v_{\lambda(1)}, \ldots, v_{\lambda(p)}), \quad \text{whenever } \varphi \in \bigwedge^p V$$

and let

$$v^\lambda = \begin{cases} v^{\lambda(1)} & \text{if } p=1, \\ v^{\lambda(1)} \wedge \cdots v^{\lambda(p)} & \text{else.} \end{cases}$$

Proposition. Suppose m is a positive integer; $p_1, \ldots, p_m$ are integers; $\phi_i \in \bigwedge^{p_i} V$, $i = 1, \ldots, m$; $v \in V$; and

$$\phi_i \llcorner v = 0, \; i = 1, \ldots, m.$$

Then

$$(\phi_1 \wedge \cdots \wedge \phi_m) \llcorner v = 0.$$

Proof. Induct on m. The assertion holds trivially if $m = 1$. Suppose that $m > 1$ and that the Proposition holds for smaller m. We use the skewderivation property of $\wedge$ and the inductive hypothesisto obtain

$$(\phi_1 \wedge \cdots \phi_m) \llcorner v = (\phi_1 \llcorner v) \wedge (\phi_2 \cdots \wedge \phi_m) + (-1)^{p_1} \phi_1 \wedge \big((\phi_2 \wedge \cdots \wedge \phi_m) \llcorner v\big) = 0.$$

□

Proposition. Suppose $p \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$ and $\lambda \in \Lambda(p, n)$. Then

$$v^\lambda \llcorner v_j = \begin{cases} (-1)^{i-1} v^{\lambda_i} & \text{if } \lambda(i) = j \text{ for some } i \in \{1, \ldots, p\}, \\ 0 & \text{else.} \end{cases}$$

Proof. If $j \notin \mathbf{rng}\, \lambda$ then $v^\lambda \llcorner v_j = 0$ by the preceding Proposition so let us suppose that $j = \lambda(i)$ for some $i \in \{1, \ldots, p\}$.

We induct on p. The Lemma holds trivially if $p = 1$. Assume that $p \in \{2, \ldots, n\}$ and that the Lemma holds for smaller p. Then

$$(-1)^{i-1}(v^\lambda \llcorner v_j) = (v^j \wedge v^{\lambda_i}) \llcorner v_j = (v^j \llcorner v_j) \wedge v^{\lambda_i} - v^j \wedge (v^{\lambda_i} \llcorner v_j) = v^{\lambda_i}$$

where we have used the inductive hypothesis to infer that $v^{\lambda_i} \llcorner v_j = 0$. □

Proposition. Suppose p in $\{1, \ldots, n\}$. Then

$$v_{[\mu]}(v^\lambda) = \delta^\mu_\lambda, \; \lambda, \mu \in \Lambda(p, n).$$

Proof. Suppose there is $i \in \{1, \ldots, p\}$ such that $\mu(i) \notin \mathbf{rng}\, \lambda$. Then

$$v_{[\mu]}(v^\lambda) = -v_{[\mu_i]}(v^\lambda \llcorner v_{\mu(i)}) = 0$$

so the Proposition holds in this case.

We show the Proposition holds in case $\mathbf{rng}\, \lambda = \mathbf{rng}\, \mu$ by induction on p. In case $p = 1$ the assertion to be proved holds trivially so suppose $p > 1$ and that the assertion to be proved holds for smaller p. Now

$$(v^{\lambda(1)} \wedge v^{\lambda_1}) \llcorner v_{\lambda(1)} = (v^{\lambda(1)} \llcorner v_{\lambda(1)}) v^{\lambda_1} - v^{\lambda(1)} \wedge (v^{\lambda_1} \llcorner v_{\lambda(1)}) = v^{\lambda_1}$$

and $\lambda = \mu$ so

$$v_{[\mu]}(v^\lambda) = v_{[\lambda]}(v^\lambda) = v_{[\lambda_1]}\big((v^{\lambda(1)} \wedge v^{\lambda_1}) \llcorner v_{\lambda(1)}\big) = v_{[\lambda_1]}(v^{\lambda_1}) = 1.$$

□

Corollary. For each $p \in \{1, \ldots, n\}$ the set $\{v^\lambda : \lambda \in \Lambda(p, n)\}$ is a basis for $\bigwedge^p V$ and has cardinality $\binom{n}{p}$. Moreover,

$$\varphi = \sum_{\lambda \in \Lambda(p,n)} v_{[\lambda]}(\varphi)\, v^\lambda \quad \text{whenever } \varphi \in \textstyle\bigwedge^p V. \tag{1}$$

Proof. Suppose $c_\lambda,\ \lambda \in \Lambda(p,n)$, are such that

$$\sum_{\lambda\in\Lambda(p,n)} c_\lambda v^\lambda = 0.$$

Let $\mu \in \Lambda(p,n)$. Applying $v_{[\mu]}$ to both sides of the preceding equation to and making use of the previous Theorem we infer that $c_\mu = 0$. It follows that $\{v^\lambda : \lambda \in \Lambda(p,n)\}$ is independent and has cardinality $\binom{n}{p}$.

That (1) holds follows from the fact that for any $\mu \in \Lambda(p,n)$ both sides have the same value on $(v_{\mu(1)},\ldots,v_{\mu(p)})$. That $\{v^\lambda : \lambda \in \Lambda(p,n)\}$ spans $\bigwedge^p V$ follows immediately from (1). $\square$

Remark. A special case of the preceding Theorem is the all important

$$\mathbf{dim} \bigwedge^n V = 1.$$

The Exterior Power of a Linear Map.

Recall that if V and W are vector spaces and $l \in \mathbf{L}(V;W)$ we have the linear map

$$\bigwedge^p l : \bigwedge^p W \to \bigwedge^p V$$

defined by setting

$$\bigwedge^p l\,(\varphi)(v_1,\ldots,v_p) = \varphi(l(v_1),\ldots,l(v_p))$$

whenever $\varphi \in \bigwedge^p W$ and $v_1,\ldots,v_p \in V$.

Theorem. Suppose $\varphi \in \bigwedge^p V$ and $\psi \in \bigwedge^q V$. Then

$$\left(\bigwedge^{p+q} l\right)(\varphi\wedge\psi) = \left(\bigwedge^p l\big(\varphi\right) \wedge \left(\bigwedge^q l\big(\psi\right).$$

Proof. Induct on $p+q$. We leave the details as an exercise for the reader. $\square$

Theorem. Suppose W and Z are vector spaces, $l : V \to W$, $m : W \to Z$ and l and m are linear. Then

$$\bigwedge^p m \circ l = \bigwedge^p l \circ \bigwedge^p m.$$

Proof. This is straight forward calculation. $\square$

The Covariant Exterior Product.

Suppose V is a vector space and $n = \mathbf{dim}\, V < \infty$. For each $v \in V$ we let $v^{**} \in V^{**}$ be such that

$$v^{**}(\omega) = \omega(v) \quad \text{whenever } \omega \in V^*$$

and note that $V \ni v \mapsto v^{**} \in V^{**}$ is a linear isomorphism from V onto V^{**}. In what follows we will frequently identify $v^{**} \in V^{**}$ with $v \in V$.

Definition. For each integer p we let

$$\bigwedge\nolimits_p V = \bigwedge^p V^*.$$

Whenever p is a integer not less than 2 we and $v_1,\ldots,v_p \in V$ we let

$$\bigwedge\nolimits_p (v_1,\ldots,v_p) = v_1\wedge\cdots\wedge v_p = v_1^{**}\wedge\cdots\wedge v_p^{**} \in \bigwedge\nolimits_p (V)$$

thereby defining

$$\bigwedge_p \in \bigwedge^p (V; \bigwedge_p (V)).$$

Note that for any integers p, q

$$\bigwedge_p V \times \bigwedge_q V \xrightarrow{\wedge} \bigwedge_{p+q} V$$

is defined.

Note that the associativity and anticommutativity of covariant exterior multiplication follow from the corresponding properties of contravariant exterior multiplication.

Bases. Suppose

$$v_1, \ldots, v_n \quad \text{and} \quad v^1, \ldots, v^n$$

are dual basic sequence for V and V^*, respectively.

Suppose $p \in \{1, \ldots, n\}$. For each λ in $\Lambda(p, n)$ let

$$v_\lambda = \begin{cases} v_{\lambda(1)} & \text{if } p = 1, \\ v_{\lambda(1)} \wedge \cdots \wedge v_{\lambda(p)} & \text{else.} \end{cases}$$

Note that

$$\{v_\lambda : \lambda \in \Lambda(p, n)\} \text{ is a basis for } \bigwedge_p V$$

and that its cardinality is

$$\binom{n}{p}.$$

For each λ in $\Lambda(p, n)$ let $v^{[\lambda]} \in \left(\bigwedge_p V\right)^*$ be such that

$$v^{[\lambda]}(v_\mu) = \delta^\lambda_\mu, \quad \mu \in \Lambda(p, n).$$

Definition. Suppose W is a finite dimensional vector space and $l \in \mathbf{L}(V; W)$. We define

$$\bigwedge_p l \in \mathbf{L}(\bigwedge_p V; \bigwedge_p W)$$

by letting

$$\bigwedge_p l = \bigwedge^p l^*.$$

Proposition. Suppose $l \in \mathbf{L}(V; W)$, p is an integer at least 2 and $v_1, \ldots, v_p \in V$. Then

$$\left(\bigwedge_p l\right)(v_1 \wedge \cdots \wedge v_p) = l(v_1) \wedge \cdots \wedge l(v_p).$$

Proof. We have

$$\begin{aligned} \bigwedge_p l(v_1 \wedge \cdots \wedge v_p) &= \bigwedge^p l^*(v_1^{**} \wedge \cdots \wedge v_p^{**}) \\ &= v_1^{**} \circ l^* \wedge \cdots \wedge v_p^{**} \circ l^* \\ &= l(v_1)^{**} \wedge \cdots \wedge l(v_p)^{**} \\ &= l(v_1) \wedge \cdots \wedge l(v_p). \end{aligned}$$

Proposition. Suppose $l \in \mathbf{L}(V; W)$, $\xi \in \bigwedge_p V$ and $\eta \in \bigwedge_p V$. Then

$$\left(\bigwedge_{p+q} l\right)(\xi \wedge \eta) = \left(\bigwedge_p l\right)(\xi) \wedge \left(\bigwedge_q\right)(\eta).$$

Proof. This is a consequence of facts already established for $\bigwedge^p$. $\square$

Proposition. Suppose $l \in \mathbf{L}(V;W)$ and $m \in \mathbf{L}(W;Z)$. Then

$$\bigwedge\nolimits_p m \circ l = \bigwedge\nolimits_p m \circ \bigwedge\nolimits_p l.$$

Proof. This is a consequence of facts already established for $\bigwedge^p *$. $\square$

The universal property of $\bigwedge_*$. Suppose p is an integer not less than 2. Then

$$\mathbf{L}(\bigwedge\nolimits_p(V);W) \ni \phi \mapsto \phi \circ \bigwedge\nolimits_p \in \bigwedge\nolimits^p(V;W)$$

carries $\mathbf{L}(\bigwedge_p;W)$ isomorphically onto $\bigwedge^p(V;W)$.

In fact, if $v_1,\ldots,v_n$ is a basis for V and $W = \mathbf{R}$ this map carries $v^{[\lambda]} = v^\lambda \circ \bigwedge_p \in \left(\bigwedge_p V\right)^*$ to $v^\lambda \in \bigwedge^p V$ for $\lambda \in \Lambda(p,n)$.

Proof. Straighforward exercise. $\square$

Remark. Taking $W = \mathbf{R}$ above we have that

$$\left(\bigwedge\nolimits_p V\right)^* \ni \phi \mapsto \phi \circ \bigwedge\nolimits_p \in \bigwedge\nolimits^p V$$

carries $\left(\bigwedge_p V\right)$ isomorphically onto $\bigwedge^p V$.

Moreover,

$$\bigwedge\nolimits_p V^* = \bigwedge\nolimits^p V^{**} \simeq \bigwedge\nolimits^p V;$$

in fact, if $\omega_1,\ldots,\omega_p \in V^*$ then $\omega_1 \wedge \cdots \wedge \omega_p \in \bigwedge_p V^*$ corresponds to $\omega_1 \wedge \cdots \wedge \omega_p \in \bigwedge^p V$ under this isomorphism.

Definition. Suppose $l \in \mathbf{L}(V;V)$. Keeping in mind that $\bigwedge_n V$ is 1-dimensional we let

$$\mathbf{det}\, l,$$

the **determinant of** l, be such that $\bigwedge_n l$ is multiplication by $\mathbf{det}\, l$.

Theorem. Suppose $l, m \in \mathbf{L}(V;V)$.

$$\mathbf{det}(l \circ m) = \mathbf{det}(l)\,\mathbf{det}(m).$$

Proof. Let $\xi \in \bigwedge_n V \sim \{0\}$. We calculate

$$\begin{aligned}
\mathbf{det}(l \circ m)\,\xi &= \left(\bigwedge\nolimits_n l \circ m\right)(\xi) \\
&= \left(\bigwedge\nolimits_n l \circ \bigwedge\nolimits_n m\right)(\xi) \\
&= \left(\bigwedge\nolimits_n l \left(\bigwedge\nolimits_n m\right)(\xi)\right) \\
&= \left(\bigwedge\nolimits_n l\right)(\mathbf{det}(m)\,\xi) \\
&= \mathbf{det}(m)\left(\bigwedge\nolimits_n l\right)(\xi) \\
&= \mathbf{det}(m)\,\mathbf{det}(l)\,\xi.
\end{aligned}$$

$\square$

Corollary. Suppose $l \in \mathbf{L}(V;V)$. Then l is invertible if and only if $\mathbf{det}\, l \neq 0$ in which case

$$\mathbf{det}\, l^{-1} = (\mathbf{det}\, l)^{-1}.$$

Corollary. Suppose W is a vector space, i carries V isomorphically onto W, $l \in \mathbf{L}(V;V)$, $k \in \mathbf{L}(W;W)$ and the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{l} & V \\ \downarrow i & & \downarrow i \\ W & \xrightarrow{k} & W. \end{array}$$

Then

$$\mathbf{det}\, k = \mathbf{det}\, l.$$

Proof. Suppose $\xi \in \bigwedge_n V$. Then

$$\begin{aligned} \mathbf{det}\, l \left(\bigwedge\nolimits_n i \right)(\xi) &= \bigwedge\nolimits_n i((\mathbf{det}\, l)\xi) \\ &= \bigwedge\nolimits_n i \left(\bigwedge\nolimits_n l(\xi) \right) \\ &= \bigwedge\nolimits_n i \circ l(\xi) \\ &= \bigwedge\nolimits_n k \circ i(\xi) \\ &= \bigwedge\nolimits_n k \left(\bigwedge\nolimits_n i(\xi) \right) \\ &= \mathbf{det}\, k \left(\bigwedge\nolimits_n i \right)(\xi). \end{aligned}$$

□

Definition. Let $\mathbf{S}_n$ be the group of permutations of $\{1, \ldots, n\}$. Let

$$\mathbf{s} : \mathbf{S}_n \to \mathbf{GL}(\mathbf{R}^n)$$

be defined by the requirement that

$$\mathbf{s}(\sigma)(\mathbf{e}_j) = \mathbf{e}_{\sigma(j)},\ \sigma \in \mathbf{S}_n,\ j = 1, \ldots, n.$$

Note that $\mathbf{s}$ is a homomorphism. We the map

$$\mathbf{sgn} = \mathbf{det} \circ \mathbf{s} : \mathbf{S}_n \to \{-1, 1\}$$

and call it the **signature** or **index**.

Theorem.

$$\mathbf{sgn}(\sigma \circ \tau) = \mathbf{sgn}(\sigma)\mathbf{sgn}(\tau),\ \sigma, \tau \in \mathbf{S}_n.$$

Moreover,

$$\mathbf{sgn}(\sigma) = \mathbf{card}\, \{(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : \sigma(i) > \sigma(j)\}$$

for any $\sigma \in \mathbf{S}_n$.

Proof. The first assertion follows directly from the product rule for determinants.

The second assertion is proved by induction on n. Observe that if $\sigma \in \mathbf{S}_n$ and $k \in \{1, \ldots, n\}$ is such that $\sigma(k) = n$ then

$$\mathbf{card}\, \{(i,n) \in \{1, \ldots, n\} : \sigma(i) > \sigma(n)\}$$

and

$$v_{\sigma(1)} \wedge \cdots v_{\sigma(n)} = (-1)^{n-i} v_{\tau(1)} \wedge \cdots \wedge v_{\tau(n-1)} \wedge v^n$$

where $\tau \in \mathbf{S}_{n-1}$ is such that

$$\tau(k) = \begin{cases} \sigma(k) & \text{if } k < i, \\ \sigma(k) - 1 & \text{if } k \geq i, \end{cases} \qquad k \in \{1, \ldots, n-1\}.$$

Inner Products.

Suppose V has an inner product $\bullet$. Let

$$\beta : V \to V^*$$

be the corresponding polarity:

$$\beta(v)(w) = v \bullet w \quad \text{whenever } v, w \in V.$$

Theorem. Suppose p is an integer not less than 2. Then there is one and only one inner product

$$\bullet$$

on $\bigwedge_p V$ such that

(1) $v_1 \wedge \cdots\cdots \wedge v_p \bullet w_1 \wedge \cdots \wedge w_p = \beta(v_1) \wedge \cdots \wedge \beta(v_p)(w_1, \ldots, w_p)$ whenever $v_i,\ w_i \in V,\ i = 1, \ldots, p.$

Moreover, if $v_1, \ldots, v_n$ is an orthonormal basic sequence for V then

$$\{v_\lambda : \lambda \in \Lambda(p, n)\}$$

is an orthonormal basis for $\bigwedge_p V$.

Proof. By virtue of earlier work we have an isomorphism

$$\bigwedge_p V \xrightarrow{\bigwedge_p \beta} \bigwedge_p V^* = \bigwedge^p V^{**} \simeq \bigwedge^p V \simeq \left(\bigwedge_p V\right)^*$$

which has the property that if $v_1, \ldots, v_n$ is an orthonormal basic sequence for V then $v_\lambda,\ \lambda \in \Lambda(p, n)$ is carried to the the member of $\left(\bigwedge_p V\right)^*$ which is dual to v_λ. The Theorem now follows easily. □

Theorem. Suppose p is an integer not less than 2 and $v_1, \ldots, v_p \in V$. Then

$$|v_1 \wedge v_2 \wedge \cdots \wedge v_p| \leq |v_1||v_2 \wedge \cdots \wedge v_p|$$

with equality only if

$$v_1 \in \mathbf{span}\,\{v_2, \ldots, v_p\}^\perp.$$

Proof. Write

$$v_1 = u_1 + w_1$$

where $u_1 \in \mathbf{span}\,\{v_2, \ldots, v_p\}$ and $w_1 \in \mathbf{span}\,\{v_2, \ldots, v_p\}^\perp$. Then, as $w_1 \in \cap_{i=2}^p \mathbf{ker}\,\beta(v_i)$,

$$\beta(v_1) \wedge \beta(v_2) \cdots \wedge \beta(v_p) \llcorner w_1 = \beta(v_1)(w_1)\beta(v_2) \wedge \cdots \wedge \beta(v_p) = |w_1|^2 \beta(v_2) \wedge \cdots \wedge \beta(v_p)$$

so

$$\begin{aligned} |v_1 \wedge v_2 \wedge \cdots \wedge v_p|^2 &= \beta(v_1) \wedge \beta(v_2) \wedge \cdots \wedge \beta(v_p)(w_1, v_2, \ldots, v_p) \\ &= (\beta(v_1) \wedge \beta(v_2) \wedge \cdots \wedge \beta(v_p) \llcorner w_1)\,(v_2, \ldots, v_p) \\ &= |w_2|^2 \beta(v_2) \wedge \cdots \wedge \beta(v_p)(v_2, \ldots, v_p) \\ &= |w_2|^2 |v_2 \wedge \cdots \wedge v_p|^2. \end{aligned}$$

□

Theorem. Suppose W is a finite dimensional inner product space and $l \in \mathbf{L}(V, W)$. Then

$$(\bigwedge_p l)^* = \bigwedge_p l^*.$$

Proof. This will come later. □