1. An extremely useful abstract closure principle.

Suppose X is a vector space over $\mathbb{R}$ and

$$|\cdot| : X \to [0,\infty]$$

is such that

(i) $|cx| = |c||x|$ whenever $c \in \mathbf{R}$ and $x \in X$;
(ii) $|x+y| \le |x| + |y|$ whenever $x, y \in X$.

(If $|x| < \infty$ for each $x \in X$ we say $|\cdot|$ is a **seminorm on** X; obviously, a norm on X is a seminorm on X.)

For each $a \in X$ and $0 < r < \infty$ let

$$\mathbf{U}(a,r) = \{x \in X : |x-a| < r\} \quad \text{and let} \quad \mathbf{B}(a,r) = \{x \in X : |x-a| \le r\}.$$

As should come as no surprise, one calls $\mathbf{U}(a,r)$ the **open ball with center** a **and radius** r and one calls $\mathbf{B}(a,r)$ the **closed ball with center** a **and radius** r.

We declare a subset U of X to be open if for each $a \in U$ there is $r \in (0,\infty)$ such that $\mathbf{U}(a,r) \subset U$. It is a simple matter which we leave to the reader to verify that the open sets are a topology on X which respect to which the open balls are open and the closed balls are closed. One easily verifies that this topology is Hausdorff if and only if

$$|x| = 0 \;\Leftrightarrow\; x = 0 \quad \text{whenever } x \in X.$$

Proposition 1.1. Suppose Y is a normed vector space, $K : X \to Y$ and K is linear. Then K is continuous linear if and only if there is $M \in [0,\infty)$ such that

$$|K(x)| \le M|x| \quad \text{whenever } x \in X. \tag{1}$$

(Here and in what follows $|\cdot|$ on the left denotes the norm on Y. This abuse of notation rarely,if ever, causes trouble.)

Proof. Suppose K is continuous. Since $K(0) = 0 \in \mathbf{U}(0,1)$ and K is continuous there is $r \in (0,\infty)$ such that $\mathbf{U}(0,r) \subset K^{-1}[\mathbf{U}(0,a)r$ which amounts to saying that

$$|K(x)| < |x| \quad \text{whenever } x \in X \text{ and } |x| < r.$$

Let $s \in (0,r)$.

Suppose $s \in X \sim \{0\}$. Then

$$\left|\frac{s}{|x|}x\right| = \frac{s}{|x|}|x| = s < r$$

so

$$|K(x)| = |K\left(\frac{|x|}{s}\left(\frac{s}{|x|}x\right)\right) = \frac{|x|}{s}K\left(\frac{s}{|x|}x\right) < \frac{|x|}{s}.$$

Moreover, $|K(0)| = |0| = 0$. Letting $s \downarrow r$ we find that (1) holds with $M = 1/r$.

It is obvious that K is continuous if (1) holds for some $M \in [0,\infty)$. □

Definition 1.1. We say Y is a **Banach space** if Y is a normed vector space which is complete with respect to the metric

$$X \times X \ni (x,y) \mapsto |x-y|$$

where $|\cdot|$ is the norm.

Let

$$W = \{w \in X : |w| < \infty\}.$$

Proposition 1.2. W is a linear subspace of X and $|\cdot||W$ is a seminorm on W.

Proof. Simple exercise for the reader. □

Theorem 1.1. Suppose

(i) U is a linear subspace of W;

(ii) Y is a Banach space;

$$l : U \to Y;$$

l is linear; $0 \le M < \infty$; and

$$|l(u)| \le M|u| \quad \text{whenever } u \in U; \tag{1}$$

(iii) V is the closure of U.

Then there is a linear function

$$L : V \to Y$$

such that

(iv) $L|V = l$;

(v) $|L(v)| \le M|v|$ whenever $v \in V$.

Moreover, if $K : V \to Y$ is a continuous function and $K|U = l$ then $K = L$.

Remark 1.1. Note that, by definition,

$$V = \{v \in W : \text{for each } r > 0 \text{ there is } u \in U \text{ such that } |v - u| < r\}.$$

It is also worth noting that

$$V = \{x \in X : \text{for each } r > 0 \text{ there is } u \in U \text{ such that } |x - u| < r\}.$$

Proof. We have

$$|l(u_1) - l(u_2)| = |l(u_1 - u_2)| \le M|u_1 - u_2| \quad \text{whenever } u_1, u_2 \in U.$$

Thus $\mathbf{Lip}(l) \le M < \infty$. By the preceding Theorem there is a function $L : V \to Y$ such that $L|U = l$ and $\mathbf{Lip}(L) = \mathbf{Lip}(l)$. (Well, not *exactly*. Do you see why?)

We proceed to show L is linear.

Suppose $v \in V$, $c \in \mathbf{R}$. For any $u \in U$ we have

$$\begin{aligned}
&|L(cv) - cL(v)| \\
&\quad = |L(cv) - l(cu) + cl(u) - cL(v)| \\
&\quad \le |L(cv) - l(cu)| + |cl(u) - cL(v)| \\
&\quad = |L(cv) - L(cu)| + |cL(u) - cL(v)| \\
&\quad = |L(cv) - L(cu)| + |c||L(u) - L(v)| \\
&\quad \le M|cv - cu| + |c|M|u - v| \\
&\quad = 2M|c||u - v|.
\end{aligned}$$

Since $|u - v|$ may be made arbitrarily small we find that $L(cv) = cL(v)$.

Suppose $v_1, v_2 \in V$. For any $u_1, u_2 \in U$ we have

$$\begin{aligned}
&|L(v_1 + v_2) - (L(v_1) + L(v_2))| \\
&\quad = |L(v_1 + v_2) - l(u_1 + u_2) - (L(v_1) - l(u_1) + L(v_2) - l(u_2))| \\
&\quad \leq |L(v_1 + v_2) - l(u_1 + u_2)| + |L(v_1) - l(u_1)| + |L(v_2) - l(u_2)| \\
&\quad = |L(v_1 + v_2) - L(u_1 + u_2)| + |L(v_1) - L(u_1)| + |L(v_2) - L(u_2)| \\
&\quad \leq M|(v_1 + v_2) - (u_1 + u_2)| + M|v_1 - u_1| + M|v_2 - u_2| \\
&\quad \leq M(|v_1 - u_1| + |v_2 - u_2|) + M|v_1 - u_1| + M|v_2 - u_2| \\
&\quad = 2M(|v_1 - u_1| + |v_2 - u_2|).
\end{aligned}$$

Since $|v_1 - u_1|$ and $|v_2 - u_2|$ may be made arbitrarily small we find that $L(v_1 + v_2) = L(v_1) + L(v_2)$.

Thus L is linear.

Finally, if $K : V \to Y$ is continuous $K|U = l$ we have that $K = L$ from earlier work. (Well, again, not *exactly*.) □