

1. AXIOMS AND RULES OF INFERENCE FOR PROPOSITIONAL LOGIC.

Suppose $\mathcal{T} = (L, A, \mathcal{R})$ is a formal theory. Whenever \mathcal{H} is a finite subset of L and $C \in L$ it is evident that

$$(\mathcal{H}, C) \in \mathcal{R} \Rightarrow \mathcal{H} \vdash C.$$

Fix a set X of propositional variables. We work with the language

$$\mathbf{p}(X).$$

1.1. **The standard setup (or so I think).** This is, essentially, what you see in the coursepack on page 78.

We axiomatize propositional logic by using following rules of inference.

Suppose A, B, C are statements. Then

$$\mathbf{EM} \quad (\emptyset, \sim A \vee A)$$

$$\mathbf{Ass} \quad (\{(A \vee (B \vee C))\}, ((A \vee B) \vee C))$$

$$\mathbf{Ex} \quad (\{A\}, (B \vee A))$$

$$\mathbf{Contr} \quad (\{(A \vee A)\}, A)$$

$$\mathbf{Cut} \quad (\{(A \vee B), (\sim A \vee C)\}, (B \vee C))$$

are rules of inference.

EM stands for “excluded middle”; **Ass** stands for “associative”; **Ex** stands for “expansion”; **Contr** stands for “contraction”; and **Cut** stands for “cut”.

An **axiom** is a rule of inference where the set of hypotheses is empty; thus **EM** is an axiom.

In addition,

$$\mathbf{And1} \quad (\emptyset, (\sim (A \wedge B) \vee \sim (\sim A \vee \sim B)))$$

$$\mathbf{And2} \quad (\emptyset, (\sim \sim (\sim A \vee \sim B) \vee (A \wedge B)))$$

$$\mathbf{Imp1} \quad (\emptyset, (\sim (A \rightarrow B) \vee (\sim A \vee B)))$$

$$\mathbf{Imp2} \quad (\emptyset, (\sim (\sim A \vee B) \vee (A \rightarrow B)))$$

$$\mathbf{Iff1} \quad (\emptyset, (\sim (A \leftrightarrow B) \vee ((A \rightarrow B) \wedge (B \rightarrow A))))$$

$$\mathbf{Iff2} \quad (\emptyset, (\sim ((A \rightarrow B) \wedge (B \rightarrow A)) \vee (A \leftrightarrow B)))$$

are rules of inference.

Note that all of the words appearing above are statements.

Proposition 1.1. Suppose

$$\mathcal{H} \vdash C$$

is one of the rules of inference for propositional logic. Then

$$\mathcal{H} \models C.$$

Proof. We need to show that if

$$T \subset X \text{ and } \mathbf{t}_H(T) = 1 \text{ for } H \in \mathcal{H} \Rightarrow \mathbf{t}_C(T) = 1.$$

We leave this as an exercise for the reader. \square

Remark 1.1. Note that if A is a statement there is a canonical way using the parse tree of A to construct a statement B which contains only the connectives \sim and \vee such that

$$\mathbf{t}_A = \mathbf{t}_B.$$

Note that another way of doing this is to let B be the disjunctive normal form of A .

Lemma 1.1. Suppose Γ is a set of statements and S_1, \dots, S_n is a proof using Γ . Then $\Gamma \models S_n$.

Proof. Either (i) $S_n \in \Gamma$ or (ii) (\mathcal{H}, S_n) is a rule of inference for some $\mathcal{H} \subset \{S_j : j < n\}$. If (i) holds it is trivial that $\Gamma \models S_n$ so suppose (ii) holds. We may suppose inductively that $\Gamma \models H$ for $H \in \mathcal{H}$. The preceding Lemma implies that $\Gamma \models S_n$. \square

Theorem 1.1. (The soundness theorem.) Suppose Γ is a set of statements and A is a statement. Then

$$\Gamma \vdash A \Rightarrow \Gamma \models A.$$

Remark 1.2. We will prove that the converse statement holds if Γ is consistent.

Proof. This follows directly from the preceding Lemma. \square

1.2. **So called “derived rules of inference”.** These are just plain theorems.

Theorem 1.2. Suppose A, B, C are statements. Then

Com $\{(A \vee B)\} \vdash (B \vee A);$

NewAss $\{((A \vee B) \vee C)\} \vdash (A \vee (B \vee C));$

Proof. See pp. 80-81 in the coursepack. \square

Theorem 1.3. Suppose A, B, C, D are statements. Then

GenAss $\{(A \vee (B \vee (C \vee D)))\} \vdash (A \vee ((B \vee C) \vee D));$

GenExp $\{(A \vee B)\} \vdash (A \vee (C \vee B));$

GenContr $\{(A \vee (B \vee B))\} \vdash (A \vee B);$

GenCut $\{(A \vee (B \vee C)), (A \vee (\sim B \vee D))\} \vdash A \vee (C \vee D);$

Proof. See pp. 91-93 in the coursepack. \square

Definition 1.1. Suppose A is a statement and Γ is a set of statements. Let

$$A \vee \Gamma = \{(A \vee S) : S \in \Gamma\}.$$

Lemma 1.2. Suppose (\mathcal{H}, C) is a rule of inference and A is a statement. Then

$$A \vee \mathcal{H} \vdash A \vee C.$$

Proof. We have $\{B\} \vdash A \vee B$ from **Ex**. This implies that the assertion to be proved holds for **EM**, **And1**, **And2**, **Imp1**, **Imp2**, **Iff1** and **Imp2**.

For **Ass**, **Ex**, **Contr** and **Cut** this amounts to the so called “generalized rules of inference” on stated and proved on pp. 91-93 of the coursepack. The rest are a straightforward exercise for the reader making use of associativity. \square

Theorem 1.4. Suppose Γ is a set of statements, A is a statement, (\mathcal{H}, C) is a rule of inference and

$$\Gamma \vdash (A \vee H) \quad \text{for } H \in \mathcal{H}.$$

Then

$$\Gamma \vdash (A \vee C).$$

Proof. This follows directly from the preceding Lemma. \square

Theorem 1.5. Suppose A, B are statements and Γ is a set of statements. Then

$$\Gamma \vdash B \Rightarrow A \vee \Gamma \vdash A \vee B.$$

Proof. Indeed, if S_1, \dots, S_n is a proof of B using Γ it follows directly from the preceding Lemma that $A \vee S_1, \dots, A \vee S_n$ is a proof of $A \vee B$ using $A \vee \Gamma$. \square

Theorem 1.6. (The deduction theorem.) Suppose Γ is a set of statements, A and B are statements and

$$\Gamma \cup \{A\} \vdash B.$$

Then

$$\Gamma \vdash (A \rightarrow B).$$

Proof. It will suffice to show $\Gamma \vdash (\sim A \vee B)$. By virtue of the preceding Theorem,

$$\sim A \vee (\Gamma \cup \{A\}) \vdash \sim A \vee B$$

so there is a proof S_1, \dots, S_n of $\sim A \vee B$ using $\sim A \vee (\Gamma \cup \{A\})$.

Let J be the set of $j \in \{1, \dots, n\}$ such that, for some \mathcal{H} , (\mathcal{H}, S_j) is a rule of inference and $\mathcal{H} \subset \{i : i < j\}$. Let $I = \{1, \dots, n\} \sim J$. Observe that if $S_j = (\sim A \vee A)$ for some $j \in \{1, \dots, n\}$ then $j \in J$ since (\emptyset, S_j) is a rule of inference. So $i \in I$ there is $C_i \in \Gamma$ such that $S_i = (\sim A \vee C_i)$. $\lambda : I \rightarrow \{1, \dots, n\}$ be increasing with range I . Then

$$C_{\lambda(1)}, \dots, C_{\lambda(m)}, S_1, \dots, S_n$$

is a proof of $(\sim A \vee B)$ using Γ . Indeed, if $i \in I$ then $\{C_i\} \vdash (\sim A \vee C_i) = S_i$. \square

2. THE ADEQUACY THEOREM, FIRST VERSION.

Proposition 2.1. Suppose A is a statement. Then

$$\{A\} \vdash \sim \sim A \quad \text{and} \quad \{\sim \sim A\} \vdash A.$$

Proof. See page 81 of the coursepack for the proof of the first one. We leave the proof of the second one as an exercise for the reader. \square

Definition 2.1. Whenever A is a statement and $I \subset X$ we let

$$A_I = \begin{cases} A & \text{if } \mathbf{t}_A(I) = 1, \\ \sim A & \text{if } \mathbf{t}_A(I) = 0. \end{cases}$$

Lemma 2.1. Suppose $I \subset X$. Then

$$(1) \quad I \vdash A_I.$$

Proof. We induct on the depth d of a parse tree of A ; note that $d \geq 2$. If $d = 2$ then A is a propositional variable p so $A_I = A$ is $p \in I$ and $A_I = \sim p$ if $p \notin I$ so the statement holds trivially.

So suppose $e \in \mathbb{N}$; $e \geq 2$; the statement holds when $d \leq e$; and the depth of a parse tree for A is $e + 1$.

Suppose $A = \sim B$. We have $I \vdash B_I$ by the inductive hypothesis. If $B_I = \sim B$ then $A_I = A = B_I$ and if $B_I = B$ then $B_I \vdash \sim \sim B = \sim A = A_I$. In either case we find that $I \vdash A_I$.

If $A \neq \sim B$ then A is one of the following:

$$(B \vee C), \quad (B \wedge C), \quad (B \rightarrow C), \quad (B \leftrightarrow C).$$

If $A = (B \vee C)$ we have

B_I	C_I	A_I
$\sim B$	$\sim C$	$\sim (B \vee C)$
$\sim B$	C	$(B \vee C)$
B	$\sim C$	$(B \vee C)$
B	C	$(B \vee C)$.

If $A = (B \wedge C)$ we have

B_I	C_I	A_I
$\sim B$	$\sim C$	$\sim (B \wedge C)$
$\sim B$	C	$\sim (B \wedge C)$
B	$\sim C$	$\sim (B \wedge C)$
B	C	$(B \wedge C)$.

If $A = (B \rightarrow C)$ we have

B_I	C_I	A_I
$\sim B$	$\sim C$	$(B \rightarrow C)$
$\sim B$	C	$(B \rightarrow C)$
B	$\sim C$	$\sim (B \rightarrow C)$
B	C	$(B \rightarrow C)$.

If $A = (B \leftrightarrow C)$ we have

B_I	C_I	A_I
$\sim B$	$\sim C$	$(B \leftrightarrow C)$
$\sim B$	C	$\sim (B \leftrightarrow C)$
B	$\sim C$	$\sim (B \leftrightarrow C)$
B	C	$(B \leftrightarrow C)$.

In all these case we may assume inductively that $I \vdash B_I$ and $I \vdash C_I$.

To complete the proof it will suffice to show that if

D	E	F
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occurs in the second to fifth line of one of these tables then

$$\{D, E\} \vdash F.$$

We leave the proof of this fact to the reader; it will be necessary to use the “derived rules of inference”.

□

Theorem 2.1. Suppose A is a tautology. Then $\vdash A$.

Proof. Let $V = \{x_1, \dots, x_n\}$ be the statement letters in A . For any subset I of V we infer from the preceding Lemma that

$$I \vdash A$$

since, as A is a tautology, $A_I = A$. In particular,

$$\{x_1, \dots, x_{n-1}\} \cup \{x_n\} \vdash A \quad \text{and} \quad \{x_1, \dots, x_{n-1}\} \cup \{\sim x_n\} \vdash A.$$

By the Deduction Theorem we infer that

$$\{x_1, \dots, x_{n-1}\} \vdash (x_n \rightarrow A) \quad \text{and} \quad \{x_1, \dots, x_{n-1}\} \vdash (\sim x_n \rightarrow A).$$

By $\{\sim\sim x_n\} \vdash x_n$ and **Cut** we infer that

$$\{x_1, \dots, x_{n-1}\} \vdash A.$$

Continuing in this way we obtain $\vdash A$.

□

Corollary 2.1. Suppose A_1, \dots, A_n and B are statements. Then

$$\{A_1, \dots, A_n\} \vdash B \Leftrightarrow \models (A_1 \wedge \dots \wedge A_n) \rightarrow B.$$

Proof. We have only to observe that

$$\{A_1, \dots, A_n\} \models B \Leftrightarrow \models ((A_1 \wedge \dots \wedge A_n) \rightarrow B).$$

□

Remark 2.1. What does $(A_1 \wedge \dots \wedge A_n)$ mean? Does it matter?

Remark 2.2. So we can dispense with a lot of the proofs using the rules of inference. Hooray!

3. THE ADEQUACY THEOREM, SECOND VERSION.

We suppose throughout this section that the set of propositional variables is countable. For the proofs see Section 3.4 of the coursepack.

Theorem 3.1. Model existence theorem. If Γ is a consistent set of statements then Γ is satisfiable.

Theorem 3.2. Suppose Γ is a set of statements and A is a statement. Then

$$\Gamma \models A \Rightarrow \Gamma \vdash A.$$

Here is a weaker version that easily follows from our previous work.

Theorem 3.3. Suppose Γ is a finite set of statements and A is a statement. Then

$$\Gamma \models A \Rightarrow \Gamma \vdash A.$$

Proof. Suppose $\Gamma = \{B_1, \dots, B_n\}$ and $\Gamma \models A$. Then $\{B_1, \dots, B_{n-1}\} \models (\sim B_n \vee A)$. Arguing inductively we infer that $\{B_1, \dots, B_{n-1}\} \vdash (\sim B_n \vee A)$. Since $\{B_n, (\sim B_n \vee A)\} \vdash A$ we conclude that $\Gamma = \{B_1, \dots, B_n\} \vdash A$. □