Independent families of random variables.

Definition. Suppose $\mathcal{X}$ is a family of random variables. We say $\mathcal{X}$ is **independent** if

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

whenever $X_1, \ldots, X_n$ are distinct members of $\mathcal{X}$ and $A_1, \ldots, A_n$ are Borel subsets of $\mathbf{R}$.

Proposition. Suppose $\mathcal{X}$ is a family of random variables. Then $\mathcal{X}$ is independent if and only if

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

whenever $X_1, \ldots, X_n$ are distinct members of $\mathcal{X}$ and $(x_1, \ldots, x_n) \in \mathbf{R}^n$.

Proof. This is really simple once you get straight what a Borel set is. We won't do this, though. $\square$

Proposition. Suppose $X_1, \ldots, X_n$ are distinct discrete random variables. Then $\{X_1, \ldots, X_n\}$ is independent if and only if

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n) \quad \text{whenever } (x_1, \ldots, x_n) \in \mathbf{R}^n. \tag{1}$$

Proof. This is long winded but simple minded. I hope you will see this.

Suppose $\{X_1, \ldots, X_n\}$ is independent and $(x_1, \ldots, x_n) \in \mathbf{R}^n$. Let $A_i = \{x_i\}$ for each $i = 1, \ldots, n$. Then

$$\begin{aligned}
p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= P(X_1 = x_1, \ldots, X_n = x_n) \\
&= P(X_1 \in A_1, \ldots, X_n \in A_n) \\
&= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\
&= P(X_1 = x_1) \cdots P(X_n = x_n) \\
&= p_{X_1}(x_1) \cdots p_{X_n}(x_n);
\end{aligned}$$

thus (1) holds.

One the other hand suppose (1) holds and $A_i \subset \mathbf{R}$, $i = 1, \ldots, n$. Then

$$\begin{aligned}
P(X_1 \in A_1, \ldots, X_n \in A_n) &= P(\cup_{x_1 \in A_1, \ldots, x_n \in A_n} \{X_1 = x_1, \ldots, X_n = x_1\}) \\
&= \sum_{x_1 \in A_1, \ldots, x_n \in A_n} P(X_1 = x_1, \ldots, X_n = x_1) \\
&= \sum_{x_1 \in A_1, \ldots, x_n \in A_n} p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \\
&= \sum_{x_1 \in A_1, \ldots, x_n \in A_n} p_{X_1}(x_1) \cdots p_{X_n}(x_n) \\
&= \left(\sum_{x_1 \in A_1} p_{X_1}(x_1) \right) \cdots \left(\sum_{x_n \in A_n} p_{X_n}(x_n) \right) \\
&= \left(\sum_{x_1 \in A_1} P(X_1 = x_1) \right) \cdots \left(\sum_{x_n \in A_n} P(X_n = x_n) \right) \\
&= (P(\cup_{x_1 \in A_1} \{X_1 = x_1)) \cdots (P(\cup_{x_n \in A_n} \{X_n = x_n)) \\
&= P(X_1 \in A_1) \cdots P(X_n \in A_n)
\end{aligned}$$

so $\{X_1, \ldots, X_n\}$ is independent. $\square$

Definition. We say the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is **continuous** if there exists a function

$$f_{\mathbf{X}} = f_{X_1,\ldots,X_n} : \mathbf{R}^n \to [0, \infty),$$

called a **(joint) probability density function**, such that

$$\begin{aligned}
F_{\mathbf{X}}(\mathbf{x}) &= F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \\
&= \int_{\mathbf{w} \le \mathbf{x}} f_{\mathbf{X}}(\mathbf{w})\, d\mathbf{w} \\
&= \int \cdots \int_{w_1 \le x_1, \ldots, w_n \le x_n} f_{X_1,\ldots,X_n}(w_1, \ldots, w_n)\, dw_1 \cdots dw_n \\
&= \int_{-\infty}^{x_n} \cdots \left(\int_{-\infty}^{x_1} f_{X_1,\ldots,X_n}(w_1, \ldots, w_n)\, dw_1 \right) \cdots dw_n.
\end{aligned}$$

We use the following formula *very* frequently.

Theorem. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is continuous random vector and R is a Borel subset of $\mathbf{R}^n$. Then

$$P(\mathbf{X} \in R) = P((X_1, \ldots, X_n) \in R) = \int_R f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_R f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Proof. This is straight forward but technical exercise which we omit. $\square$

Corollary. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is continuous random vector and $1 \le i_1 < \cdots < i_m \le n$. Then $(X_{i_1}, \ldots, X_{i_m})$ is a continuous random vector for which

$$f_{X_{i_1},\ldots,X_{i_m}}(x_{i_1}, \ldots, x_{i_m})$$

equals the integral over all of the other variables.

Definition. Suppose R is a Borel set in $\mathbf{R}^n$. We say the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is **uniformly distributed over** R if

$$P(\mathbf{X} \in Q) = \frac{|Q \cap R|}{|R|} \quad \text{whenever } Q \text{ is a Borel subset of } \mathbf{R}^n.$$

It is a straightforward but technical exercise which we omit to show that $\mathbf{X}$ is continuous with pdf given by

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \frac{1}{|R|} & \text{if } \mathbf{x} \in R, \\ 0 & \text{else.} \end{cases}$$