Problem 74, page 113. In successive rolls of a pair of fair dice, what is the probability of getting 2 sevens before 6 even numbers?

Let A, B and C be the events that you get a seven, and even number, or neither on a given roll, respectively. Thus

$$P(A) = \frac{6}{36} = \frac{1}{6}, \quad P(B) = \frac{18}{36} = \frac{1}{2}, \quad P(C) = 1 - P(A) - P(B) = \frac{1}{3}.$$

We have

$$\begin{aligned}
&P(2 \text{ sevens before } 6 \text{ evens}) \\
&\qquad = \sum_{i=0}^{5} P(i \text{ evens before } 2 \text{ sevens}) \\
&\qquad = \sum_{i=0}^{5} \sum_{j=0}^{i} P(i \text{ evens before } 2 \text{ sevens} | j \text{ evens before first seven}) P(j \text{ evens before first seven}) \\
&\qquad = \sum_{i=0}^{5} \sum_{j=0}^{i} P(i - j \text{ evens before first seven}) P(j \text{ evens before first seven}).
\end{aligned}$$

For each nonnegative integer j let E_j be the event that you roll j evens before the first seven. We have

$$\begin{aligned}
P(E_0) &= P(E_0 | \text{seven on the first roll}) P(\text{seven on the first roll}) \\
&\quad + P(E_0 | \text{even on the first roll}) P(\text{even on the first roll}) \\
&\quad + P(E_0 | \text{neither seven nor even on the first roll}) P(\text{neither seven nor even on the first roll}) \\
&= 1\, P(A) + 0\, P(B) + P(E_0) P(C)
\end{aligned}$$

so

$$P(E_0) = \frac{P(A)}{1 - P(C)} = \frac{1}{4}.$$

For $j > 0$ we have

$$\begin{aligned}
P(E_j) &= P(E_j | \text{seven on the first roll}) P(\text{seven on the first roll}) \\
&\quad + P(E_j | \text{even on the first roll}) P(\text{even on the first roll}) \\
&\quad + P(E_j | \text{neither seven nor even on the first roll}) P(\text{neither seven nor even on the first roll}) \\
&= 0\, P(A) + P(E_{j-1}) P(B) + P(E_j) P(C)
\end{aligned}$$

which implies that

$$(1 - P(C)) P(E_j) = P(E_{j-1}) P(B)$$

so

$$P(E_j) = \left(\frac{P(B)}{1 - P(C)} \right)^j P(E_0) = \left(\frac{3}{4} \right)^j \frac{1}{4}.$$

So the desired probability is

$$\sum_{i=0}^{5} \sum_{j=0}^{i} \left(\frac{3}{4} \right)^{i-j} \frac{1}{4} \left(\frac{3}{4} \right)^j \frac{1}{4} = (\frac{1}{4})^2 \sum_{j=0}^{i} (i+1) \left(\frac{3}{4} \right)^i \simeq .5550.$$