Transformations of continuous random vectors.

Theorem. Suppose

(1) U is an open subset of $\mathbf{R}^n$ and $\mathbf{X} = (X_1, \ldots, X_n)$ is a continuous random vector with range U;

(2) V is an open subset of $\mathbf{R}^n$ and

$$\mathbf{g} = (g_1, \ldots, g_n) : U \to V$$

is a continuously differentiable mapping carrying U one-to-one onto V with continuously differentiable inverse and

(3) $\mathbf{Y} = \mathbf{g}(\mathbf{X})$.

Then $\mathbf{Y}$ is continuous and

$$f_{\mathbf{Y}}(\mathbf{y}) \;=\; \begin{cases} f_{\mathbf{X}}(\mathbf{x})\,|\mathbf{det}\,\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x})|^{-1} & \text{if } \mathbf{x} \in U \text{ and } \mathbf{y} = \mathbf{g}(\mathbf{x}), \\ 0 & \text{else.} \end{cases}$$

Proof. Suppose $\mathbf{x} \in U$ and $\mathbf{y} = \mathbf{g}(\mathbf{x})$. For each positive h let

$$C_h \;=\; (y_1 - h, y_1 + h) \times \cdots \times (y_n - h, y_n + h).$$

Then

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \lim_{h \downarrow 0} \frac{P(\mathbf{Y} \in C_h)}{(2h)^n} \\ &= \lim_{h \downarrow 0} \frac{P(\mathbf{X} \in \mathbf{g}^{-1}(C_h))}{(2h)^n} \\ &= \lim_{h \downarrow 0} \frac{1}{(2h)^n} \int_{\mathbf{g}^{-1}(C_h)} f_{\mathbf{X}}(\mathbf{w})\, d\mathbf{w} \\ &= f_{\mathbf{X}}(\mathbf{x}) \lim_{h \downarrow 0} \frac{1}{(2h)^n} \int_{\mathbf{g}^{-1}(C_h)} d\mathbf{w} \\ &= f_{\mathbf{X}}(\mathbf{x})\,|\mathbf{det}\,\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x})|^{-1}. \end{aligned}$$