Test Two Mathematics 135.02 Fall 2009

Answer Key.

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

I have neither given nor received aid in the completion of this test.
Signature:

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		Your Score
1	10 pts.	
2	10 pts.	
3	10 pts.	
4	35 pts.	
5	15 pts.	
6	15 pts.	
7	15 pts.	
8	20 pts.	
Total	130 pts.	

1. 10 pts. Suppose X_1, X_2 are independent random variables with means μ_1, μ_2 and variances σ_1^2, σ_2^2, respectively. Compute $\mathrm{E}((X_1 + 3X_2)^2)$.

Solution. For $i = 1, 2$ we have $\mathrm{E}(X_i^2) = \mathrm{Var}(X_1) + \mathrm{E}(X_i)^2 = \sigma_i^2 + \mu_i^2$. Also, as X_1 and X_2 are independent, we have $\mathrm{E}(X_1X_2) = \mathrm{E}(X_1)\mathrm{E}(X_2) = \mu_1\mu_2$. Thus

$$\begin{aligned}\mathrm{E}((X_1 + 3X_2)^2) &= \mathrm{E}(X_1^2 + 6X_1X_2 + 9X_2^2)\\ &= \mathrm{E}(X_1^2) + 6\mathrm{E}(X_1X_2) + 9\mathrm{E}(X_2^2)\\ &= \sigma_1^2 + \mu_1^2 + 6\mu_1\mu_2 + 9\sigma_2^2 + 9\mu_2^2.\end{aligned}$$

2. 10 pts. The random vector (X, Y) is uniformly distributed on the triangle with vertices $(0, 0), (1, 0), (0, 1)$. Compute $\mathrm{E}(XY)$.

Solution. Let T be the triangle with vertices $(0, 0), (1, 0), (0, 1)$; then $T = \{(x, y) \in \mathbb{R}^2 : 0 < y < 1 - x\}$ and T has area $1/2$. Thus

$$f_{X,Y} = \begin{cases} 2 & \text{if } 0 < y < 1 - x, \\ 0 & \text{else.} \end{cases}$$

so that

$$E(XY) = 2\int_T xy\,dxdy = 2\int_0^1 \left(\int_0^{1-x} xy\,dy\right)\,dx = \frac{1}{12}.$$

3. 10 pts. Joe can jump over a wall with probability 4/5. Use the Central Limit Theorem to estimate the probability that in one hundred attempts Joe jumps over the wall at least ninety times. Assume that success on one jump is independent of success on any other jump.

(The independence assumption is obviously unrealistic if if Joe doesn't rest between jumps or if Joe waits so long between jumps that he gets old!)

Solution. Let S_{100} be the number of times Joe jumps over the wall. Let $\mu = 4/5$ and let $\sigma = \sqrt{\mu(1-\mu)} = \sqrt{4/25} = 2/5$. Also $\sqrt{100} = 10$. We have

$$\begin{aligned}
\mathrm{P}(90 \le S_{100}) &= \mathrm{P}(89.5 \le S_{100}) \\
&= \mathrm{P}\left(\frac{89.5 - 100(4/5)}{10(2/5)} \le \frac{S_{100} - 100(4/5)}{10(2/5)}\right) \approx 1 - \Phi\left(\frac{89.5 - 100(4/5)}{10(2/5)}\right) \\
&= 1 - \Phi(2.375) \\
&\approx 1 - .9912 \\
&= .0088.
\end{aligned}$$

4. The random vector (X, Y) is continuous and, for some $C > 0$, has a density given by

$$f_{X,Y}(x,y) = C\begin{cases} xy & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{else.} \end{cases}$$

(i) **(5 pts.)** Determine C.
(ii) **(10 pts.)** Determine if X and Y are independent;
(iii) **(20 pts.)** Calculate the variance of XY^2.

Solution. For (i) we want

$$1 = C\int_0^1 \left(\int_0^1 xy\,dx\right)\,dy = \frac{C}{4}$$

so $C = 4$.

For (ii) we have

$$f_{X,Y}(x,y) = g(x)g(y) \quad \text{where} \quad g(x) = \begin{cases} 2x & \text{if } 0 < x < 1, \\ 0 & \text{else} \end{cases}$$

so X and Y are independent.

For (iii) we note that, as $\int_0^1 g(x)\,dx = 1$, we have $f_X = g = f_Y$.

Since X and Y are independent we have

$$\mathrm{E}((XY^2)^2) = \mathrm{E}(X^2Y^4) = \mathrm{E}(X^2)\mathrm{E}(Y^4) = \left(\int_0^1 x^2(2x)\,dx\right)\left(\int_0^1 y^4(2y)\,dy\right) = \frac{1}{6}$$

and

$$\mathrm{E}(XY^2) = \left(\int_0^1 x(2x)\,dx\right)\left(\int_0^1 y^2(2y)\,dy\right) = \frac{1}{3}.$$

Thus

$$\mathrm{Var}(XY^2) = \mathrm{E}((XY^2)^2) - (\mathrm{E}(XY^2))^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

5. 15 pts. An urn contains b black balls, w white balls and r red balls. A ball is drawn from the urn and a fair coin is flipped $w+r$ times if the ball is black; $b+r$ times if the ball is white; and $b+w$ times if the ball is red. Calculate the expected number of heads.

Solution. Let B, W, R be the events that a black,white,red ball is drawn from the urn, respectively. Let N be the number of heads. Then

$$\begin{aligned}\mathrm{E}(N) &= \mathrm{E}(N|B)\mathrm{P}(B) + \mathrm{E}(N|W)\mathrm{P}(W) + \mathrm{E}(N|R)\mathrm{P}(R)\\ &= \frac{w+r}{2}\frac{b}{b+w+r} + \frac{b+r}{2}\frac{w}{b+w+r} + \frac{b+w}{2}\frac{r}{b+w+r}\\ &= \frac{bw+br+wr}{b+w+r}.\end{aligned}$$

6. 15 pts. The random variable X is normally distributed with mean μ and variance σ^2. Moreover,

$$\mathrm{P}(X < 3) = .8413 \quad \text{and} \quad \mathrm{P}(X < 5) = .9772.$$

Determine μ and σ.

Solution. Note that

$$\Phi(1) = .8413 \quad \text{and} \quad \Phi(2) = .9772.$$

Also, $\frac{(X-\mu)}{\sigma}$ is normally distributed with mean μ and variance σ^2.

We have

$$.8413 = \mathrm{P}(X < 3) = \mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{3-\mu}{\sigma}\right)$$

and

$$.9772 = \mathrm{P}(X < 5) = \mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right)$$

so

$$\frac{3-\mu}{\sigma} = 1 \quad \text{and} \quad \frac{5-\mu}{\sigma} = 2.$$

Solving this system of two linear equations in the unknowns μ and σ we find that $\mu = 1$ and $\sigma = 2$.

7. 15 pts. Suppose N is a random variable with values in the nonnegative integers such that $\mathrm{E}(N) = 1$ and $\mathrm{Var}(N) = 1$. Use Chebychev's Inequality to show that

$$\mathrm{P}(N > n) \le \frac{1}{n^2} \quad \text{for any positive integer } n.$$

Solution. Note that

$$\{N > n\} = \{N = n+1\} \subset \{|N-1| \ge n\} = \{|N - \mathrm{E}(N)| \ge n\}$$

so, by Chebychev's Inequality,

$$\mathrm{P}(N > n) \le \mathrm{P}(|N - \mathrm{E}(N)| \ge n) \le \frac{\mathrm{Var}(N)}{n^2} = \frac{1}{n^2}.$$

8. 20 pts. Suppose $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables, all with expectation p. Let T be the number of *transitions* which is, by definition, the number of i with $2 \le i \le n$ such that $X_{i-1} \ne X_i$. Compute the expectation of T. (Hint: Use indicator random variables.)

Solution. Let $q = 1 - p$. For each $i = 2, \ldots, n$ let Y_i be the indicator of the event

$$\{X_{i-1} \ne X_i\} = \{X_{i-1} = 1, X_i = 0\} \cup \{X_{i-1} = 0, X_i = 1\}$$

so that $\mathrm{E}(Y_i) = pq + qp = 2pq$. Since $T = \sum_{i=2}^{n} Y_i$ we find that $\mathrm{E}(T) = 2(n-1)pq$.