## Answer Key.

## TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

I have neither given nor received aid in the completion of this test. Signature:

|  |  | Your Score |
| :---: | :---: | :---: |
| 1 | 10 pts. |  |
| 2 | 10 pts. |  |
| 3 | 10 pts. |  |
| 4 | 35 pts. |  |
| 5 | 15 pts. |  |
| 6 | 15 pts. |  |
| 7 | 15 pts. |  |
| 8 | 20 pts. |  |
| Total | 130 pts. |  |

1. 10 pts. Suppose $X_{1}, X_{2}$ are independent random variables with means $\mu_{1}, \mu_{2}$ and variances $\sigma_{1}^{2}, \sigma_{2}^{2}$, respectively. Compute $\mathrm{E}\left(\left(X_{1}+3 X_{2}\right)^{2}\right)$.

Solution. For $i=1,2$ we have $\mathrm{E}\left(X_{i}^{2}\right)=\operatorname{Var}\left(X_{1}\right)+\mathrm{E}\left(X_{i}\right)^{2}=\sigma_{i}^{2}+\mu_{i}^{2}$. Also, as $X_{1}$ and $X_{2}$ are independent, we have $\mathrm{E}\left(X_{1} X_{2}\right)=\mathrm{E}\left(X_{1}\right) \mathrm{E}\left(X_{2}\right)=\mu_{1} \mu_{2}$. Thus

$$
\begin{aligned}
\mathrm{E}\left(\left(X_{1}+3 X_{2}\right)^{2}\right) & =\mathrm{E}\left(X_{1}^{2}+6 X_{1} X_{2}+9 X_{2}^{2}\right) \\
& =\mathrm{E}\left(X_{1}^{2}\right)+6 \mathrm{E}\left(X_{1} X_{2}\right)+9 \mathrm{E}\left(X_{2}^{2}\right) \\
& =\sigma_{1}^{2}+\mu_{1}^{2}+6 \mu_{1} \mu_{2}+9 \sigma_{2}^{2}+9 \mu_{2}^{2}
\end{aligned}
$$

2. 10 pts. The random vector $(X, Y)$ is uniformly distributed on the triangle with vertices $(0,0),(1,0),(0,1)$. Compute $\mathrm{E}(X Y)$.

Solution. Let $T$ be the triangle with vertices $(0,0),(1,0),(0,1)$; then $T=\{(x, y) \in$ $\left.\mathbb{R}^{2}: 0<y<1-x\right\}$ and $T$ has area $1 / 2$. Thus

$$
f_{X, Y}= \begin{cases}2 & \text { if } 0<y<1-x \\ 0 & \text { else }\end{cases}
$$

so that

$$
E(X Y)=2 \int_{T} x y d x d y=2 \int_{0}^{1}\left(\int_{0}^{1-x} x y d y\right) d x=\frac{1}{12}
$$

3. 10 pts. Joe can jump over a wall with probability $4 / 5$. Use the Central Limit Theorem to estimate the probability that in one hundred attempts Joe jumps over the wall at least ninety times. Assume that success on one jump is independent of success on any other jump.
(The independence assumption is obviously unrealistic if if Joe doesn't rest between jumps or if Joe waits so long between jumps that he gets old!)

Solution. Let $S_{100}$ be the number of times Joe jumps over the wall. Let $\mu=4 / 5$ and let $\sigma=\sqrt{\mu(1-\mu)}=\sqrt{4 / 25}=2 / 5$. Also $\sqrt{100}=10$. We have

$$
\begin{aligned}
\mathrm{P}\left(90 \leq S_{100}\right) & =\mathrm{P}\left(89.5 \leq S_{100}\right) \\
& =\mathrm{P}\left(\frac{89.5-100(4 / 5)}{10(2 / 5)} \leq \frac{S_{100}-100(4 / 5)}{10(2 / 5)}\right) \approx 1-\Phi\left(\frac{89.5-100(4 / 5)}{10(2 / 5)}\right) \\
& =1-\Phi(2.375) \\
& \approx 1-.9912 \\
& =.0088
\end{aligned}
$$

4. The random vector $(X, Y)$ is continuous and, for some $C>0$, has a density given by

$$
f_{X, Y}(x, y)=C \begin{cases}x y & \text { if } 0<x<1 \text { and } 0<y<1 \\ 0 & \text { else }\end{cases}
$$

(i) (5 pts.) Determine $C$.
(ii) (10 pts.) Determine if $X$ and $Y$ are independent;
(iii) (20 pts.) Calculate the variance of $X Y^{2}$.

Solution. For (i) we want

$$
1=C \int_{0}^{1}\left(\int_{0}^{1} x y d x\right) d y=\frac{C}{4}
$$

so $C=4$.
For (ii) we have

$$
f_{X, Y}(x, y)=g(x) g(y) \quad \text { where } \quad g(x)= \begin{cases}2 x & \text { if } 0<x<1 \\ 0 & \text { else }\end{cases}
$$

so $X$ and $Y$ are independent.
For (iii) we note that, as $\int_{0}^{1} g(x) d x=1$, we have $f_{X}=g=f_{Y}$.
Since $X$ and $Y$ are independent we have

$$
\mathrm{E}\left(\left(X Y^{2}\right)^{2}\right)=\mathrm{E}\left(X^{2} Y^{4}\right)=\mathrm{E}\left(X^{2}\right) \mathrm{E}\left(Y^{4}\right)=\left(\int_{0}^{1} x^{2}(2 x) d x\right)\left(\int_{0}^{1} y^{4}(2 y) d y\right)=\frac{1}{6}
$$

and

$$
\mathrm{E}\left(X Y^{2}\right)=\left(\int_{0}^{1} x(2 x) d x\right)\left(\int_{0}^{1} y^{2}(2 y) d y\right)=\frac{1}{3}
$$

Thus

$$
\operatorname{Var}\left(X Y^{2}\right)=\mathrm{E}\left(\left(X Y^{2}\right)^{2}\right)-\left(\mathrm{E}\left(X Y^{2}\right)\right)^{2}=\frac{1}{6}-\left(\frac{1}{3}\right)^{2}=\frac{1}{18}
$$

5. 15 pts. An urn contains $b$ black balls, $w$ white balls and $r$ red balls. A ball is drawn from the urn and a fair coin is flipped $w+r$ times if the ball is black; $b+r$ times if the ball is white; and $b+w$ times if the ball is red. Calculate the expected number of heads.

Solution. Let $B, W, R$ be the events that a black, white, red ball is drawn from the urn, respectively. Let $N$ be the number of heads. Then

$$
\begin{aligned}
\mathrm{E}(N) & =\mathrm{E}(N \mid B) \mathrm{P}(B)+\mathrm{E}(N \mid W) \mathrm{P}(W)+\mathrm{E}(N \mid R) \mathrm{P}(R) \\
& =\frac{w+r}{2} \frac{b}{b+w+r}+\frac{b+r}{2} \frac{w}{b+w+r}+\frac{b+w}{2} \frac{r}{b+w+r} \\
& =\frac{b w+b r+w r}{b+w+r}
\end{aligned}
$$

6. 15 pts. The random variable $X$ is normally distributed with mean $\mu$ and variance $\sigma^{2}$. Moreover,

$$
\mathrm{P}(X<3)=.8413 \quad \text { and } \quad \mathrm{P}(X<5)=.9772
$$

Determine $\mu$ and $\sigma$.
Solution. Note that

$$
\Phi(1)=.8413 \quad \text { and } \quad \Phi(2)=.9772
$$

Also, $\frac{(X-\mu)}{\sigma}$ is normally distributed with mean $\mu$ and variance $\sigma^{2}$.
We have

$$
.8413=\mathrm{P}(X<3)=\mathrm{P}\left(\frac{X-\mu}{\sigma}<\frac{3-\mu}{\sigma}\right)
$$

and

$$
.9772=\mathrm{P}(X<5)=\mathrm{P}\left(\frac{X-\mu}{\sigma}<\frac{5-\mu}{\sigma}\right)
$$

so

$$
\frac{3-\mu}{\sigma}=1 \quad \text { and } \quad \frac{5-\mu}{\sigma}=2
$$

Solving this system of two linear equations in the unknowns $\mu$ and $\sigma$ we find that $\mu=1$ and $\sigma=2$.
7. 15 pts . Suppose $N$ is a random variable with values in the nonnegative integers such that $\mathrm{E}(N)=1$ and $\operatorname{Var}(N)=1$. Use Chebychev's Inequality to show that

$$
\mathrm{P}(N>n) \leq \frac{1}{n^{2}} \quad \text { for any positive integer } n
$$

Solution. Note that

$$
\{N>n\}=\{N=n+1\} \subset\{|N-1| \geq n\}=\{|N-\mathrm{E}(N)| \geq n\}
$$

so, by Chebychev's Inequality,

$$
\mathrm{P}(N>n) \leq \mathrm{P}(|N-\mathrm{E}(N)| \geq n) \leq \frac{\operatorname{Var}(N)}{n^{2}}=\frac{1}{n^{2}}
$$

8. 20 pts. Suppose $X_{1}, X_{2}, \ldots, X_{n}$ are independent Bernoulli random variables, all with expectation $p$. Let $T$ be the number of transitions which is, by definition, the number of $i$ with $2 \leq i \leq n$ such that $X_{i-1} \neq X_{i}$. Compute the expectation of $T$. (Hint: Use indicator random variables.)

Solution. Let $q=1-p$. For each $i=2, \ldots, n$ let $Y_{i}$ be the indicator of the event

$$
\left\{X_{i-1} \neq X_{i}\right\}=\left\{X_{i-1}=1, X_{i}=0\right\} \cup\left\{X_{i-1}=0, X_{i}=1\right\}
$$

so that $\mathrm{E}\left(Y_{i}\right)=p q+q p=2 p q$. Since $T=\sum_{i=2}^{n} Y_{i}$ we find that $\mathrm{E}(T)=2(n-1) p q$.

