Test Two   Mathematics 135.01   Fall 2007

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

I have neither given nor received aid in the completion of this test.
Signature:

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Problem	Grade
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

The average was 90.5 and the standard deviation was 28.1.

1. 1pts. Suppose X is normal with mean 3 and variance 4. Determine a, b such that $aX + b$ is normal with mean 5 and variance 9.

Suppose $0 < a < \infty$ and $b \in \mathbb{R}$. We have

$$5 = E(aX + b) = aE(X) + b = 3a + b$$

and

$$9 = \text{Var}(aX + b) = a^2\text{Var}(X) = 4a^2$$

so $a = 3/2$ and $b = 1/2$. Note that this works for not matter what the distribution of X is.

2. 10 pts. Suppose X is exponential distributed with parameter $\lambda = 2$. Calculate the pdf of $\sqrt{X}$.

Let $Y = \sqrt{X}$ and note that range of Y is $(0, \infty)$. For $0 < y < \infty$ we have

$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = 1 - e^{-2y^2}.$$

Differentiating, we obtain

$$f_Y(y) = \begin{cases} 0 & \text{if } -\infty < y \le 0, \\ 4ye^{-2y^2} & \text{if } 0 < y < \infty. \end{cases}$$

3. 20 pts. Suppose X uniform on $(-1, 2)$. Calculate the pdf of $X^2 + 1$.

Let $Y = X^2$. The range of Y is $[1, 5)$. For $1 \le y \le 2$ we have

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^2 + 1 \le y) \\ &= P(-\sqrt{y-1} \le X \le \sqrt{y-1}) \\ &= \frac{2\sqrt{y-1}}{3} \end{aligned}$$

and if $2 < y < 5$ we have

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^2 + 1 \le y) = P(-1 \le X \le \sqrt{y-1}) \\ &= \frac{1 + \sqrt{y-1}}{3}. \end{aligned}$$

Differentiating, we obtain

$$f_Y(y) = \begin{cases} 0 & \text{if } y \le 1, \\ \frac{1}{3\sqrt{y-1}} & \text{if } 1 \le y \le 2, \\ \frac{1}{6\sqrt{y-1}} & \text{if } 2 < y < 5, \\ 0 & \text{if } 5 \le y. \end{cases}$$

4. 20 pts. Suppose (X, Y) is uniformly distributed on $(0, 1) \times (0, 2)$, A is a random variable such that

$$P(A = a) = \begin{cases} \frac{1}{4} & \text{if } a = 0, \\ \frac{3}{4} & \text{if } a = 1 \end{cases}$$

and

$$Z = \begin{cases} X & \text{if } A = 0, \\ Y & \text{if } A = 1. \end{cases}$$

Calculate the cdf of Z and determined if Z is continuous. Calculate the expectation of Z. (Hint: Condition on A. Consider $\{Z \le z\}$ for $z \le 0$, $0 < z < 1$ $1 \le z < 2$, $2 \le z$.

Suppose $0 < z < 1$. Then

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(Z \le z|A=0)P(A=0) + P(Z \le z|A=1)P(A=1) \\ &= P(X < z)\frac{1}{4} + P(Y < z)\frac{3}{4} \\ &= \frac{z}{4} + \frac{3z}{8}. \end{aligned}$$

Suppose $1 \le z < 2$. Then

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(Z \le z|A=0)P(A=0) + P(Z \le z|A=1)P(A=1) \\ &= \frac{1}{4} + P(Y < z)\frac{3}{4} \\ &= \frac{1}{4} + \frac{3z}{8}. \end{aligned}$$

Thus

$$F_Z(z) = \begin{cases} 0 & \text{if } z \le 0, \\ \frac{z}{4} + \frac{3z}{8} & \text{if } 0 < z < 1, \\ \frac{1}{4} + \frac{3z}{8} & \text{if } 1 \le z < 2, \\ 1 & \text{if } 2 \le z. \end{cases}$$

Since F_Z is continuous we find that Z is continuous and

$$f_Z(z) = \begin{cases} 0 & \text{if } z \le 0, \\ \frac{5}{8} & \text{if } 0 < z < 1, \\ \frac{3}{8} & \text{if } 1 \le z < 2, \\ 0 & \text{if } 2 \le z \end{cases}$$

so

$$E(Z) = \frac{5}{8}\int_0^1 z\,dz + \frac{3}{8}\int_1^2 z\,dz = \frac{7}{8}.$$

5. 10 pts. Suppose (X, Y) is uniformly distributed on $\Omega = \{(x, y) \in \mathbf{R}^2 : 0 < y < x^2 < 1\}$. Calculate $E(XY)$. (For this one I want a number for answer which means you have to do whatever integrals are necessary.)

The area of Ω is

$$\int_{-1}^{1}\left(\int_0^{x^2} dy\right) dx = \frac{2}{3}$$

so

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{2} & \text{if } (x, y) \in \Omega, \\ 0 & \text{if } (x, y) \notin \Omega. \end{cases}$$

Thus

$$E(XY) = \int\int_{\Omega} xy\, f_{X,Y}\, dxdy = \frac{3}{2}\int_{-1}^{1}\left(\int_0^{x^2} xy\, dy\right) dx = 0.$$

6. 10 pts. Suppose $W_1, W_2, \ldots, W_n, \ldots$ are independent continuous random variables each of which is exponentially distributed with parameter $\lambda = 2$. For each $n = 1, 2, \ldots$ let $T_n = \sum_{m=1}^{n} W_m$ and let

$$N = \#\{n : 3 < T_n < 13\}.$$

Determine $P(N = n)$, $n = 0, 1, 2, \ldots$. (This is easy given what's in the book and what Prof. Huber did in class. Or look in the book in the right place.)

N is Poisson with parameter $(13 - 3)\lambda = 26$ so

$$P(N = n) = \begin{cases} e^{-26}\frac{(26)^n}{n!} & \text{for } n = 0, 1, 2, \ldots, \\ 0 & \text{else.} \end{cases}$$

7. 20 pts. Suppose (X, Y, Z) is uniformly distributed on $(0,1) \times (0,1) \times (0,1)$. Calculate $P(X < \min\{Y, Z\})$.

Let $A = \{(x, y, z) \in (0,1)^3 : x < \min\{y, z\}\}$. If $0 < x < 1$ then the area of

$$\{(y, z) \in (0,1)^2 : x < \min\{y, z\}\} = \{(y, z) \in (0,1)^2 : x < y \text{ and } x < z\}$$

is $(1 - x)^2$ so the desired probability is

$$\text{volume}(A) = \int_0^1 (1 - x)^2\, dx = \frac{1}{3}.$$

8. 10 pts. A certain machine has five components C_i, $i = 1, \ldots, 5$ whose failure times are exponentially distributed with parameters λ_i, $i = 1, \ldots, 5$, respectively. The machine functions if C_1, C_2, C_3 function or if C_3, C_4, C_5 function. Calculate the expected time of failure of the machine.

Let $Z_1 = \min\{C_1, C_2\}$, let $Z_2 = \min\{C_4, C_5\}$, let $Z_3 = \max\{Z_1, Z_2\}$ and let $Z = \min\{C_3, Z_3\}$. So Z is the time of failure of the machine. Suppose

$0 < z < \infty$. Note that Z_1 and Z_2 are exponential with parameters $\lambda_1 + \lambda_2$ and $\lambda_4 + \lambda_5$. Then

$$\begin{aligned} P(Z > z) &= P(C_3 > z)P(Z_3 > z) \\ &= P(C_3 > z)(1 - P(Z_3 \le z) \\ &= P(C_3 > z)(1 - P(Z_1 \le z)P(Z_2 \le z) \\ &= e^{-\lambda_3 z}(1 - (1 - e^{-(\lambda_1+\lambda_2)z})(1 - e^{-(\lambda_4+\lambda_5)z})) \\ &= e^{-(\lambda_1+\lambda_2+\lambda_3)z} + e^{-(\lambda_1+\lambda_4+\lambda_5)z} - e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)z} \end{aligned}$$

so

$$\begin{aligned} E(Z) &= \int_0^\infty P(Z > z)\, dz \\ &= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_1+\lambda_4+\lambda_5} - \frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5}. \end{aligned}$$

9. 20 pts. I have a machine that makes widgets. Let E_i be the event that the ith widget produced is acceptable. Assume that $P(E_i) = p$ for all $i = 1, 2, 3, \ldots$ and some $p \in (0, 1)$; assume also that the events E_i, $i = 1, 2, 3, \ldots$ are independent.

Suppose N is a large integer and $0 < w < 1$. How many widgets does the machine have to produce so that I can be $100w\%$ sure that I get at least N acceptable widgets? (I want you to use the normal approximation; your answer will depend on that z such that $\Phi(z) = w$.)

For each $i = 1, 2, \ldots$ let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th widget is acceptable,} \\ 0 & \text{if the } i\text{th widget is unacceptable.} \end{cases}$$

For each $n = 1, 2, \ldots$ let $S_n = \sum_{i=1}^n X_i$. Thus S_n is number of acceptable widgets out of a lot of n widgets. Let $q = 1 - p$ and suppose Z is standard normal. Then

$$\begin{aligned} w &< P(S_n \ge N) \\ &= P\left(\frac{N - np}{\sqrt{npq}} \le \frac{S_n - np}{\sqrt{npq}}\right) \\ &\approx 1 - \Phi\left(\frac{N - np}{\sqrt{npq}}\right) \\ &= \Phi\left(\frac{np - N}{\sqrt{npq}}\right). \end{aligned}$$

Now

$$
\begin{aligned}
&\Leftrightarrow w < \Phi\left(\frac{np-N}{\sqrt{npq}}\right)\\
&\Leftrightarrow \Phi^{-1}(w) < \frac{np-N}{\sqrt{npq}}\\
&\Leftrightarrow z < \frac{np-N}{\sqrt{npq}}\\
&\Leftrightarrow \frac{np-N}{\sqrt{npq}} > z\\
&\Leftrightarrow np - \sqrt{npq}z - N > 0\\
&\Leftrightarrow \sqrt{n} > \frac{z\sqrt{pq} + \sqrt{z^2pq+4pN}}{2p}\\
&\Leftrightarrow n > \frac{N}{p} + \frac{z}{2p}\left(qz + \sqrt{q(z^2q+4N)}\right).
\end{aligned}
$$

10. 20 pts. Suppose (X, Y) is uniform on $(0,1)\times(0,1)$. Show that XY is continuous and calculate it's pdf. (This is a bit tricky.)

We could use the change of variables formula to calculate $f_{X,XY}$ and then use the fact that

$$f_{XY}(z) = \int_{-\infty}^{\infty} f_{X,XY}(x,z)\,dz.$$

Instead, we'll do it from first principles by calulating the cdf of XY and then differentiating.

Note that the range of $Z = XY$ is $(0,1)$. If $0 < z < 1$ then

$$
\begin{aligned}
F_Z(z) &= P(Z \le z)\\
&= P(XY \le z)\\
&= \text{area}(\{(x,y) \in (0,1)\times(0,1) : xy \le z\})\\
&= z + \int_z^1 \frac{z}{x}\,dx \quad \text{Draw a picture!}\\
&= z + z\ln x\Big|_{x=z}^{x=1}\\
&= z(1-\ln z).
\end{aligned}
$$

Differentiating, we obtain

$$F_Z(z) = \begin{cases} 0 & \text{if } z \le 0;\\ -\ln z & \text{if } 0 < z < 1;\\ 1 & \text{if } 1 \le z. \end{cases}$$

11. pts. Suppose $W_1, W_2, \ldots, W_n, \ldots$ are independent continuous random variables each of which has the same distribution. For each $n = 1, 2, \ldots$ let $T_n = \sum_{m=1}^{n} W_m$. Suppose m are positive integers and $0 < a < b < \infty$. Calculate $P(bT_m < aT_{2m})$.

Suppose m is a positive integer and $0 < a < b < \infty$. Calculate $P(bT_m < aT_{2m})$.

We have

$$\{bT_m < aT_{2m}\} = \left\{T_m < \frac{a}{b-a}(T_{2m} - T_m)\right\}$$

and note that T_m and $T_{2m} - T_m$ are independent random variables with the same distribution.

So suppose $0 < c < \infty$ and X, Y are continuous independent random variables with the same cdf F and pdf f. Then

$$\begin{aligned} P(X < cY) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{cx} f(x) f(y)\, dy \right) dx \\ &= \int_{-\infty}^{\infty} f(x) F(cx)\, dx \end{aligned}$$

and this is as far as you can take it. In case $c = 1$ we have

$$\int_{-\infty}^{\infty} f(x) F(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2} \frac{d}{dx} F(x)^2\, dx = \frac{1}{2}.$$

(Ok, I admit it, I thought you could do the above in the same way even when $c \neq 1$.)