Test One   Mathematics 135.02   Fall 2009

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

**I have neither given nor received aid in the completion of this test.
Signature:**

_____

		Your Score
1	10 pts.	
2	10 pts.	
3	15 pts.	
4	20 pts.	
5	10 pts.	
6(i)	10 pts.	
6(ii)	15 pts.	
6(iii)	20 pts.	
7	15 pts.	
8	15 pts.	
Total	140 pts.	

1. 10 pts. A, B, C, D are distinct mutually independent events. Compute $P((A \cup D) \sim (B \sim C))$ in terms of $P(A), P(B), P(C), P(D)$.

Solution. Let Ω be the sample space. Since $A \cup D$ and $\Omega \sim (B \sim C)$ are independent, since A and D are independent and since B and $\Omega \sim C$ are independent we have

(1)
$$P((A \cup D) \sim (B \sim C) = P((A \cup D))P(\Omega \sim (B \sim C))$$
$$= (P(A) + P(D) - P(A \cap D))(1 - P(B \cap (\Omega \sim C)))$$
$$= (P(A) + P(D) - P(A)P(D))(1 - (P(B)(1 - P(C))))$$

2. 10 pts. A friend of yours tells you that of the 30 graduate students in history at Duke, 20 speak French, 15 speak German and 10 speak Italian. Moreover, he says, 10 speak French and German, 8 speak French and Italian and 3 speak all three languages. How would you explain to your friend that he cannot be right?

Solution. Let F, G, I be the sets of the students that speak French, German, Italian, respectively. Let N be the set of the students that speak none of these.

(When I gave you this problem I intended that N be empty. So be it.) By the Inclusion-Exclusion Principle we have

$$
\begin{aligned}
30 &= |F \cup G \cup I \cup N| \\
&= |F| + |G| + |I| + |N| - (|F \cap G| + |F \cap I| + |G \cap I|) + |F \cap G \cap I| \\
&= 20 + 15 + 10 - (10 + 8 + |G \cap I|) + 3 \\
&= 30 - |G \cap I| + |N|
\end{aligned}
$$

so

$$
|G \cap I| = |N|.
$$

Were $|N| = 0$ we would have $|G \cap I| = 0$ which would be impossible since

$$
3 = |F \cap G \cap I| \leq |G \cap I|.
$$

So suppose $|N| > 0$. Using the fact that $|A \sim B| = |A| - |A \cap B|$ for sets A and B we find that

(2)
$$
|(F \cap G) \sim I| = 10 - 3 = 7, \quad |(F \cap I) \sim G| = 8 - 3 = 5, \quad |(G \cap I) \sim F| = |N| - 3.
$$

Using the fact that

$$
|A \sim (B \cup C)| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|
$$

for finite sets A, B, C we find that

(3)
$$
\begin{aligned}
|F \sim (G \cup I)| &= 20 - 10 - 8 + 3 = 5, \\
|G \sim (I \cup F)| &= 15 - |N| - 10 + 3 = 8 - |N|, \\
|I \sim (F \cup G)| &= 10 - 8 - |N| + 3 = 5 - |N|.
\end{aligned}
$$

These three numbers in (2), the three numbers in (3), together with $|F \cap G \cap I|$ and $|N|$ should sum to 30 but they sum to 27.

3. 15 pts. A certain set X is the disjoint union of subsets A, B, C containing $10, 8, 6$ members, respectively. How many subsets D of X are there such that $|A \cap D| \geq 9$, $1 \leq |B \cap D| \leq 2$ and $|C \cap D| = 0$? **Solution.**

$$
\left(\binom{10}{9} + \binom{10}{10} \right) \left(\binom{8}{1} + \binom{8}{2} \right).
$$

4. 20 pts. Consider N urns $U_1, \ldots, U_N$ such that U_i contains i black balls and $N - i$ white balls. An urn is picked at random and a ball is drawn from it. What is the probability the j-th urn was chosen given that the ball drawn was white? (I want a squeaky clean answer for this. You will need the formula $\sum_{j=1}^{k} = k(k+1)/2$.)

Solution. Let B_i, $i = 1, \ldots, N$, be the event that Urn i was chosen and let W be the event that a white ball was drawn. By Bayes' formula and the fact that the

$P(B_i) = 1/N$ we have

$$
\begin{aligned}
P(B_j|W) &= \frac{P(W|B_j)P(B_j)}{\sum_{i=1}^{N} P(W|B_i)P(B_i)} \\
&= \frac{P(W|B_j)}{\sum_{i=1}^{N} P(W|B_i)} \\
&= \frac{N-j}{\sum_{i=1}^{N} N-i} \\
&= \frac{N-j}{N^2 - N(N+1)/2} \\
&= \frac{2(N-j)}{N(N-1)}
\end{aligned}
$$

5. 10 pts. The continuous random variable X is such that

$$
f_X(x) = C \begin{cases} x-3 & \text{if } 4 < x < 5, \\ 0 & \text{if } x < 4 \text{ or } 5 < x. \end{cases}
$$

Determine C.

Solution. We have

$$
1 = C \int_{-\infty}^{\infty} f_X(x)\, dx = C \int_{4}^{5} (x-3)\, dx = C\frac{3}{2}
$$

so $C = 2/3$.

6. 35pts. X is a continuous random variable such that

$$
f_X(x) = \begin{cases} \frac{|x|}{10} & \text{if } -2 < x < 4, \\ 0 & \text{if } x < -2 \text{ or } 4 < x. \end{cases}
$$

(i) (10 pts.) Calculate $P(0 < X < 1)$.
(ii) (15 pts.) Calculate $P(|X| > 1)$.
(iii) (20 pts.) Explain why $X^2 + 1$ is continuous and calculate its probability density function (pdf).

Solution. For (i),

$$
P(0 < X < 1) = \int_{0}^{1} f_X(x)\, dx = \int_{0}^{1} \frac{|x|}{10}\, dx = \frac{1}{10} \int_{0}^{1} x\, dx = \frac{1}{20}.
$$

For (ii) note that $\{x \in \mathbb{R} : |x| > 1 \text{ and } -2 < x < 4\} = (-2, -1) \cup (1, 4)$. Thus

$$
P(|X| > 1) = \left(\int_{-2}^{-1} + \int_{1}^{4} \right) \frac{|x|}{10}\, dx = \int_{-2}^{-1} \frac{-x}{10}\, dx + \int_{1}^{4} \frac{x}{10}\, dx = \frac{9}{10}.
$$

For (iii) we let $Y = X^2 + 1$ and we calculate

$$F_Y(y) = P(Y \le y) = P(X^2 + 1 \le y)$$

$$= \begin{cases} 0 & \text{if } y < 1, \\ P(\sqrt{y-1} \le X \le \sqrt{y-1}) & \text{if } 1 \le y \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 1, \\ F_X(\sqrt{y-1}) - F_X(-\sqrt{y-1}) & \text{if } 1 \le y \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 1, \\ F_X(\sqrt{y-1}) - F_X(-\sqrt{y-1}) & \text{if } 1 \le y < 5, \\ F_X(\sqrt{y-1}) & \text{if } 5 \le y. \end{cases}$$

where we have used the fact that $P(X = x) = 0$ for any $x \in \mathbb{R}$ since X is continuous. Now for any $y \ge 1$ we have

$$-2 < -\sqrt{y-1} < 4 \Leftrightarrow y < 5$$

and

$$-2 < \sqrt{y-1} < 4 \Leftrightarrow y < 17.$$

Thus

$$f_Y(y) = \begin{cases} 0 & \text{if } y < 1, \\ \frac{d}{dy}\left(F_X(\sqrt{y-1}) - F_X(-\sqrt{y-1})\right) & \text{if } 1 < y < 5, \\ \frac{d}{dy}\left(F_X(\sqrt{y-1})\right) & \text{if } 5 < y < 17, \\ 0 & \text{if } 17 < y \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 1, \\ \frac{\sqrt{y-1}}{10}\frac{1}{2\sqrt{y-1}} - \left(-\frac{\sqrt{y-1}}{10}\frac{1}{2\sqrt{y-1}}\right) & \text{if } 1 < y < 5, \\ \frac{\sqrt{y-1}}{10}\frac{1}{2\sqrt{y-1}} & \text{if } 5 < y < 17, \\ 0 & \text{if } 17 < y \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 1, \\ \frac{1}{10} & \text{if } 1 < y < 5, \\ \frac{1}{20} & \text{if } 5 < y < 17, \\ 0 & \text{if } 17 < y \end{cases}$$

7. 15 pts. A fair six sided die is thrown twice. Let X_1 and X_2 be the outcomes of the first and second throw, respectively. Let $X = \min\{X_1, X_2\}$. Calculate the probability mass function (pmf) of X. (I suggest you picture the Cartesian product $D \times D$ where $D = \{1, 2, 3, 4, 5, 6\}$.)

Solution. Let $D = \{1, 2, 3, 4, 5, 6\}$. It should be clear that the range of X is D. Note that

$$P(X_1 = x_1, X_2 = x_2) = \frac{1}{6}\frac{1}{6} = \frac{1}{36} \quad \text{whenever } x_1, x_2 \in D.$$

Suppose $x \in R$. Then the event $\{X = x\}$ is the union of the

$$1 + (6 - x) + (6 - x) = 13 - 2x$$

disjoint events

$$\{X_1 = x, X_2 = x\};$$
$$\{X_1 = x, X_2 = y\}, \quad y \in R \text{ and } x < y;$$
$$\{X_1 = y, X_2 = x\}, \quad y \in R \text{ and } x < y$$

each of which has probability $1/36$. Thus

$$p_X(x) = P(X = x) = \begin{cases} 13 - 2x & \text{if } x \in R, \\ 0 & \text{else.} \end{cases}$$

Since this last function has integral unity we infer that Y is continuous.

8. 15 pts. X and Y are random variable on the same probability space such that (X, Y) is uniform on the rectangle $(2, 5) \times (3, 8)$. Calculate $P(Y < X)$.

Solution. Let $R = (2, 5) \times (3, 8)$ and let $S = \{(x, y) \in \mathbb{R}^2 : y < x\}$. Then, as (X, Y) is uniform on R,

$$P(Y < X) = \frac{\text{area}(R \cap S)}{\text{area} R}.$$

Now $R \cap S$ is the open triangle with vertices at $(3, 3), (3, 5), 5, 5)$ which has area 2 and R has area $3 \cdot 5 = 15$. Thus

$$P(Y < X) = \frac{2}{15}.$$