Test One Mathematics 135.01 Fall 2007

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

I have neither given nor received aid in the completion of this test.
Signature:

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The average was 82 and the standard deviation was 12.0949.

1. 5 pts. Suppose X is a random variable with variance 5. Compute $\mathrm{Var}(3X+9)$.

Solution.

$$\mathrm{Var}(3X+9) = \mathrm{Var}(3X) = 3^2\mathrm{Var}(X) = 3^2(5) = 45.$$

2. 5 pts. Suppose X and Y are independent variables with expectations 3 and 4, respectively. Compute $E(XY)$.

Solution. Since X and Y are independent we have

$$E(XY) = E(X)E(Y) = 3 \cdot 4 = 12.$$

3. 10 pts. How many 12 letter strings can be made with 3 A's, 4 B's and 5 C's?

Solution. From the multinomial formula we obtain

$$\binom{3+4+5}{3,\ 4,\ 5} = \frac{12!}{3!4!5!} = 27720.$$

4. 10 pts. A fair six sided die is thrown 3 times. Describe a sample space for this experiment and compute the probability that the sum of the three numbers is five.

Solution. A sample space could be $\Omega = \{1,2,3,4,5,6\}^3$ with probability $P(E) = |E|/|\Omega| = |E|/216$. The event in question is

$$E = \{(1,1,3),(1,3,1),(3,1,1),(1,2,2),(2,1,2),(2,2,1)\}$$

so that

$$P(E) = \frac{|E|}{|\Omega|} = \frac{6}{216} = \frac{1}{36}.$$

5. 20 pts. Suppose X is a random variable such that

$$P(X=1) = \frac{1}{8}, \quad P(X=3) = \frac{1}{2}, \quad P(X=5) = \frac{3}{8}.$$

Three balls are drawn from and urn containing X black balls and three white balls. Let B be the event that two of the three balls are black. Compute $P(X=5|B)$.

Solution. We will From Bayes' Formula we obtain

$$P(X=5|B)=\frac{P(B|X=5)P(X=5)}{P(B|X=1)P(X=1)+P(B|X=3)P(X=3)+P(B|X=5)P(X=5)}.$$

We have

$$\begin{aligned}
P(B|X=1)&=0;\\
P(B|X=3)&=\frac{\binom{3}{2}\binom{3}{1}}{\binom{3+3}{3}}=\frac{9}{20};\\
P(B|X=5)&=\frac{\binom{5}{2}\binom{3}{1}}{\binom{5+3}{3}}=\frac{15}{28}
\end{aligned}$$

so the answer is

$$\frac{\frac{15}{28}\frac{1}{8}}{\frac{9}{20}\frac{1}{2}+\frac{15}{28}\frac{1}{8}}=\frac{25}{109}.$$

6. 15 pts. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent identically distributed random variables such that

$$E(X_i)=4 \quad \text{and} \quad \mathrm{Var}(X_i)=4,\ i=1,2,\ldots.$$

Let

$$S=\sum_{i=1}^{100} X_i.$$

Use the Central Limit Theorem to approximate

$$P(S>425).$$

(If you do it correctly the arithmetic is simple.)

Solution. For each $i=1,2,\ldots$ we have $100E(X_i)=400$ and $\sqrt{100}\sqrt{\mathrm{Var}(X_i)}=10\cdot 2=20$. Using the normal approximation we obtain

$$\begin{aligned}
P(S_{100}>425)&=P(424.5<S_{100})\\
&=P\left(\frac{424.5-400}{20}<\frac{S_{100}-400}{20}\right)\\
&=P\left(1.225<\frac{S_{100}-400}{20}\right)\\
&\approx 1-\Phi(1.225)\\
&\approx 1-\frac{\Phi(1.23)+\Phi(1.22)}{2}\\
&\approx 1-\frac{.8907+.8888}{2}\\
&=0.11025
\end{aligned}$$

7. 15 pts. Let

$$Q=\{(x,y)\in\mathbf{R}^2 : x \text{ and } y \text{ are integers, } x\ge 0,\ y\ge 0 \text{ and } x+y\le 2\}.$$

(I suggest you draw a picture of Q.)

There are random variables X and Y such that

$$p_{X,Y}(x,y) = \begin{cases} \frac{x+y}{8} & \text{if } (x,y) \in Q, \\ 0 & \text{else.} \end{cases}$$

Calculate the mean and variance of $X+Y$ and determine if X and Y are independent.

Solution. The range of $X+Y$ is $\{0,1,2\}$. Moreover,

$$P(X+Y=1) = P(X=0,Y=1) + P(X=1,Y=0) = \frac{0+1}{8} + \frac{1+0}{8} = \frac{1}{4}$$

and

$$P(X+Y=2) = P(X=0,Y=2) + P(X=1,Y=2) + P(X=2,Y=0) = \frac{0+2}{8} + \frac{1+1}{8} + \frac{2+0}{8} = \frac{3}{4}.$$

Thus

$$E(X+Y) = \sum_z zP(X+Y=z) = 0P(X+Y=0) + 1P(X+Y=1) + 2P(X+Y=2) = \frac{1}{4} + 2\frac{3}{4} = \frac{7}{4}$$

and

$$E((X+Y)^2) = \sum_z z^2 P(X+Y=z) = 0^2 P(X+Y=0) + 1^2 P(X+Y=1) + 2^2 P(X+Y=2) = \frac{1}{4} + 4\frac{3}{4} = \frac{13}{4}.$$

In particular,

$$\operatorname{Var}(X+Y) = E((X+Y)^2) - E(X+Y)^2 = \frac{13}{4} - \left(\frac{7}{4}\right)^2 = \frac{3}{16}.$$

X and Y are not independent since $P(X=2,Y=2)=0$ whereas $P(X=2)P(Y=2) = (1/4)(1/4) \neq 0$.

8. 20 pts. Suppose A, B, C, D are independent events. Compute

$$P((A \cup B) \cap (C \cup D)) \quad \text{and} \quad P((A \sim B) \cup (C \sim D))$$

in terms of $P(A), P(B), P(C), P(D)$.

Solution. As $A \cup B$ and $C \cup D$ are independent we find that

$$\begin{aligned} &P((A \cup B) \cap (C \cup D)) \\ &\qquad = P(A \cup B)P(C \cup D) \\ &\qquad = (P(A) + P(B) - P(A \cap B))(P(C) + P(D) - P(C \cap D)) \\ &\qquad = (P(A) + P(B) - P(A)P(B))(P(C) + P(D) - P(C)P(D)). \end{aligned}$$

As $A \sim B$ and $C \sim D$ are independent we find that

$$\begin{aligned} &P((A \sim B) \cap (C \sim D)) \\ &\qquad = P(A \sim B)P(C \sim) \\ &\qquad = (P(A) - P(A \cap B))(P(C) - P(C \cap D)) \\ &\qquad = (P(A) - P(A)P(B))(P(C) - P(C)P(D)). \end{aligned}$$