1. The sum of independent normals.

Suppose X_1 and X_2 are independent normal random variables with mean 0 and variance σ_1^2 and σ_2^2, respectively. We have

$$\begin{aligned} f_{X_1+X_2}(z) &= f_{X_1} * f_{X_2}(z) \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(z-y)^2/2\sigma_1^2} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-y^2/2\sigma_2^2}\, dy \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} e^{-[\tau_1^2(y-z)^2+\tau_2^2 z^2]/2}\, dy \end{aligned}$$

where we have set

$$\tau_1^2 = \frac{1}{\sigma_1^2} \quad \text{and} \quad \tau_2^2 = \frac{1}{\sigma_2^2}.$$

Now if $a > 0$ and $b \in \mathbb{R}$ and

$$y = \frac{w}{\sqrt{a}} - \frac{b}{2a}$$

we have

$$ay^2 + by + c = a\left(\frac{w^2}{a} - \frac{wb}{\sqrt{a}} + \frac{b^2}{4a^2}\right) + b\frac{w}{\sqrt{a}} + c = w^2 + c - \frac{b^2}{4a}.$$

Letting

$$a = \tau_1^2 + \tau_2^2 \quad \text{and} \quad b = -2\tau_1^2 z,$$

it follows that

$$\begin{aligned} \tau_1^2(y-z)^2 + \tau_2^2 z^2 &= (\tau_1^2+\tau_2^2)y^2 - 2\tau_1^2 yz + \tau_1^2 z^2 \\ &= w^2 + \tau_1 z^2 + \frac{(-2\tau_1^2 z)^2}{4(\tau_1^2+\tau_2^2)} \\ &= w^2 + z^2 \frac{\tau_1^2\tau_2^2}{\tau_1^2+\tau_2^2} \\ &= w^2 + \frac{z^2}{\sigma_1^2+\sigma_2^2}. \end{aligned}$$

Putting it all together we conclude that X_1+X_2 is normal with mean 0 and variance $\sigma_1^2+\sigma_2^2$.