Convolutions and Fourier Transforms.

Definition. Suppose

$$f : \mathbf{R} \to \mathbf{R}.$$

We say f is **summable** if f is integrable over each bounded interval (whatever that means) and

$$||f|| = \int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Definition. Suppose f and g are summable. Then

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy$$

is defined and continous for all x (althought this is not immediately obvious) and $f * g$ is summable. In fact

$$\begin{aligned} ||f * g|| &= \int_{-\infty}^{\infty} |f * g(x)|\, dx \\ &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \right| dx \\ &\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x-y)g(y)|\, dy \right) dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x-y)g(y)|\, dx \right) dy \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(w)g(y)|\, dw \right) dy \quad \text{(substitute } w+y \text{ for } x\text{)} \\ &= \left(\int_{-\infty}^{\infty} |f(w)|\, dw \right) \left(\int_{-\infty}^{\infty} |g(y)|\, dy \right) \\ &= ||f||||g||. \end{aligned}$$

Proposition. Suppose f, g, h are summable and a, b are scalars. Then

(1) $f * g = g * f$;

(2) $(af + bg) * h = a(f * h) + b(g * h)$;

(3) $f * (g * h) = (f * g) * h$.

Moreover,

$$(f * g)' = f' * g \quad \text{provided } f' \text{ is summable.} \tag{4}$$

Definition. Suppose f is summable. We let

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx \quad \text{whenever } \xi \in \mathbf{R}$$

and call this function the **Fourier transform of** f; the Fourier transform of f is continuous and decays to 0 at infinity but need not be summable.

The Fourier inversion formula. Suppose f is summable. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{ix\xi} \hat{f}(\xi)\, d\xi \quad \text{for almost all } \xi \in \mathbf{R}.$$

Proof. We will give a sketch of the proof of this all important formula later. $\square$

Proposition. Suppose f and g are summable. Then

$$f \hat{*} g = \hat{f}\hat{g}.$$

Proof. Suppose $\xi \in \mathbf{R}$. Then

$$\begin{aligned}
f \hat{*} g(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi} f * g(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{-ix\xi} \left(\int_{-\infty}^{\infty} f(x-y)g(y)\, dy \right) dx \\
&= \int_{-\infty}^{\infty} e^{-ix\xi} \left(\int_{-\infty}^{\infty} f(x-y)g(y)\, dx \right) dy \\
&= \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} e^{-ix\xi} f(x-y)\, dx \right) dy \\
&= \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} e^{-i(w+y)\xi} f(w)\, dw \right) dy \quad \text{(substitute } w+y \text{ for } x\text{)} \\
&= \left(\int_{-\infty}^{\infty} e^{-iw\xi} f(w)\, dw \right) \left(\int_{-\infty}^{\infty} e^{-iy\xi} g(y)\, dy \right) \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$