.
The Strong Law of Large Numbers.

Lemma 0.1. Suppose Y is a nonnegative random variable and $1 \leq p < \infty$. Then $E(Y) \leq E(Y^p)^{1/p}$.

Proof. If $q = p/(p-1)$ we infer from the convexity of the exponential function

$$b \leq \frac{1}{p} b^p + \frac{1}{q} \quad \text{whenever } 0 < b < \infty.$$

This implies

$$\frac{Y}{J} \leq \frac{1}{p}\left(\frac{Y^p}{J^p}\right) + \frac{1}{q}$$

for any positive J. Taking expectation we find that

$$\frac{E(Y)}{J} \leq \frac{1}{p}\frac{E(Y^p)}{J^p} + \frac{1}{q}.$$

Letting $J = E(Y^p)^{1/p}$ we find that

$$\frac{E(Y)}{E(Y^p)} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

□

Theorem 0.1. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables;

$$\mu = E(|X_i|) < \infty \quad \text{for } i = 1, 2, 3, \ldots;$$

and

$$G = \left\{ \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right\}.$$

Then

$$P(G = 1).$$

Proof. We will prove this under the *additional but unnecessary* assumption that

$$K = E\left(X_i^4\right) < \infty \quad \text{for } i = 1, 2, 3, \ldots.$$

From the Lemma we find that

$$E(|X_i|^j) \leq E(X_i^4)^{4/j} < \infty \quad \text{for } j = 1, 2, 3. \tag{1}$$

Replacing X_i by $X_i - \mu$ we may assume that $E(X_i) = 0$ for $i = 1, 2, 3, \ldots$.

Let $S_n = \sum_{i=1}^n X_i$ for each $n = 1, 2, 3, \ldots$. I claim that

$$E\left(\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right) = \sum_{n=1}^{\infty} E\left(\frac{S_n^4}{n^4}\right) < \infty. \tag{2}$$

To see this, note that S_n^4 is the sum of the terms

$$T_\alpha = \frac{4!}{\alpha_1!\alpha_2!\cdots\alpha_n!} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$$

corresponding to 4-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers such that $\sum_{i=1}^n \alpha_i = 4$. Now here is the key point: Since the expectation of the product of independent random variables is the product of their expectations, we find that $E(T_\alpha) = 0$ if one or more of the α_i equal 1. Since $E(X_i) = 0$ for $i = 1, 2, 3, \ldots$ we find that *either* (i) exactly one of the α_i equals 4 and the rest are 0 *or* (ii) exactly two of the α_i are 2 and the rest are 0. There are n terms as in 1 and $n(n-1)/2$ terms as in (ii). Hence, by (1),

$$E(S_n^4) = nE(X_1^4) + \frac{n(n-1)}{2}E(X_1^2)^2 \leq \left(n + \frac{n(n-1)}{2}\right)K$$

so (2) holds.

But (2) implies that the event

$$F = \left\{\frac{\sum_{n=1}^{\infty} S_n^4}{n^4 < \infty}\right\}$$

has probability 1. Since $F \subset G$ we infer that G has probability 1. $\square$