Random vectors.

Fix a positive integer n. We say a subset R of $\mathbf{R}^n$ is a **rectangle** if it it the n-fold Cartesian product of intervals of real numbers.

Definition. Suppose $(S, \mathcal{E}, P)$ We say

$$\mathbf{X} : S \to \mathbf{R}^n$$

is a **random vector** if

$$\{\mathbf{X} \in R\} \in \mathcal{E} \qquad \text{whenever } R \text{ is a rectangle.}$$

Suppose $\mathbf{X}$ is a random vector. Its **(joint) cumulative distribution function**

$$F_{\mathbf{X}} : \mathbf{R}^n \to [0,1]$$

is defined by

$$F_{\mathbf{X}}(\mathbf{x}) = P(X_1 \leq x_1, \ldots, X_n \leq x_n) \qquad \text{for } \mathbf{x} \in \mathbf{R}^n.$$

We say $\mathbf{X}$ is **discrete** if

$$\sum_{\mathbf{x} \in \mathbf{R}^n} P(\mathbf{X} = \mathbf{x}) = 1$$

in which case its **(joint) probability mass function**

$$p_{\mathbf{X}} : \mathbf{R}^n \to [0,1]$$

is defined by

$$p_{\mathbf{X}}(\mathbf{x}) = P(\mathbf{X} = \mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathbf{R}^n.$$

We say $\mathbf{X}$ is **(jointly)(absolutely) continuous** if there is a function

$$f_{\mathbf{X}} : \mathbf{R}^n \to [0, \infty),$$

called the **(joint) probability density function** of $\mathbf{X}$, such that

$$P(\mathbf{X} \in R) = \int_R f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

whenever R is a rectangle in $\mathbf{R}^n$.

Proposition. Suppose $\mathbf{X}$ is a continuous random vector. Then

$$P(\mathbf{X} \in G) = \int_G f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

whenever G is a subset of $\mathbf{R}^n$ with well defined Jordan content.

Proof. This is technical. I'll draw some pictures that I hope will make it intuitively clear. $\square$

Example. Suppose X is uniformly distributed on $(0,1)$ and $\mathbf{X} = (X, X)$. Then $\mathbf{X}$ is neither discrete nor continuous. More generally, imagimagine choosing a random position on a curve in $\mathbf{R}^2$ or $\mathbf{R}^3$ or on a surface in $\mathbf{R}^3$.

Theorem. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is a random vector. If $X_1, \ldots, X_n$ are continous mutually independent random variables then $\mathbf{X}$ is continuous and

$$f_{\mathbf{X}} = f_{X_1}(x_1) \cdots f_{X_n}(x_n) \qquad \text{whenever } \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

If $\mathbf{X}$ is continuous and there exist functions $f_i : \mathbf{R} \to [0, \infty),\ i = 1, \ldots, n$ such that

$$f_{\mathbf{X}}(\mathbf{x}) = f_1(x_1) \cdots f_n(x_n) \qquad \text{whenever } \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$$

then $X_1, \ldots, X_n$ are continous mutually independent random variables and

$$f_{X_i} = f_i,\ i = 1, \ldots, n.$$

Proof. All you have to do is unwrap definitions. $\square$

Theorem. Suppose $X_1, \ldots, X_n$ are continous mutually independent random variables. Let

$$Z = \sum_{i=1}^{n} X_i.$$

Then Z is continuous and

$$f_Z = f_{X_1} * \cdots * f_{X_n}.$$

Proof. We'll assume that $n = 2$; one handles the general case in a manner that is obviously analagous.
Set $X = X_1$ and $Y = X_2$. For each $z \in \mathbf{R}$ set

$$B_z = \{(x, y) \in \mathbf{R}^2 : x + y \le z\}.$$

Then

$$F_X(z) = P(Z \le z) = \int\int_{B_z} f_{(X,Y)}(x, y)\, dx\, dy = \int\int_{B_z} f_X(x)\, f_Y(y)\, dx\, dy.$$

In this last integral make the substitution

$$(x, y) = T(u, v) = (u, v - u), \qquad (u, v) \in \mathbf{R}^2,$$

noting that T carries

$$A_z = \{(u, v) : v \le z\}$$

one to one and onto B_z. Thus this last integral equals

$$\int\int_{A_z} f_X(u), f_Y(v - u)\, du\, dv = \int_{-\infty}^{z} (\int_{-\infty}^{\infty} f_X(u), f_Y(v - u)\, du)\, dv.$$

Applying d/dz we obtain

$$f_Z(z) = F_Z'(z) = \int_{-\infty}^{\infty} f_X(u)\, f_Y(z - u)\, du = f_X * f_Y\, (z).$$

$\square$