Random variables.

Definition. We say

$$F : \mathbf{R} \to [0, 1]$$

is a **cumulative distribution function** if

(i) $x < y \Rightarrow \ F(x) \leq F(y)$;

(ii)$\lim_{x\downarrow -\infty} F(x) = 0$

(iii) $\lim_{x\downarrow a} F(x) = F(a)$ whenever $a \in \mathbf{R}$; and

(iv) $lim_{x\uparrow\infty} F(x) = 1$.

We say

$$p : \mathbf{R} \to [0, 1]$$

is a **probability mass function** if

(i) $\{x \in \mathbf{R} : p(x) \neq 0\}$ is countable and

(ii) $\sum_{x\in\mathbf{R}} p(x) = 1$.

We say

$$f : \mathbf{R} \to [0, \infty)$$

is a **probability density function** if

(i) f is integrable and

(iii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Let

$$(S, \mathcal{E}, P)$$

be a probability space.

Definition. We say X is a **random variable** if

$$X : S \to \mathbf{R}$$

and

$$\{s \in S : X(s) \in I\} \in \mathcal{E} \quad \text{whenever } I \text{ is an interval.}$$

Definition. Suppose X is a random variable. We let

$$F_X : \mathbf{R} \longrightarrow [0, 1],$$

be such that

$$F_X(x) = P(X \leq x) \quad \text{for } x \in \mathbf{R}.$$

Note that F_X is a cumulative distribution function which we call the **cumulative distribution function (cdf)** of X.

We say X is **discrete** if there is countable set C such that $P(\{s \in S : X(s) \notin C\}) = 0$ in which case we let

$$p_X : \mathbf{R} \to [0, 1]$$

be such that

$$p_X(x) = P(X = x) \quad \text{for } x \in \mathbf{R}.$$

Note that p_X is a probability mass function which we call the **probability mass function (pmf) of** X. Note that

$$F_X(x) = \sum_{v \leq x} p_X(v) \quad \text{for } x \in \mathbf{R}.$$

Note that X is discrete if the range of X is either finite or countably infinite.

We say X is **continuous** if there is an integrable function

$$f_X : \mathbf{R} \to [0, \infty)$$

such that

$$F_X(x) = \int_{-\infty}^{x} f_X(v)\ dv \quad \text{for } x \in \mathbf{R}.$$

Evidently, Note that f_X is a probability density function which call a **probability density function (pdf) for** X.

Suppose X_1 is a random variable associated to $(S_1, \mathcal{E}_\infty, P_1)$ and X_2 is a random variable associated to $(S_2, \mathcal{E}_\in, P_2)$. We say X_1 and X_2 **have the same distribution** if

$$F_{X_1} = F_{X_2}.$$

It is clear this notion carries over to any number of random variables. We say a set $\mathcal{X}$ of random variables is **identically distributed** if any two random variables in $\mathcal{X}$ have the same distribution.