p. 113, n. 60. There is a 50-50 chance the queen carries the gene for hemophilia. If she is a carrier, then each prince has a 50-50 chance of having hemophilia. If the queen has had three princes without the disease, what is the probability the queen is a carrier? If there is a fourth prince, what is the probability that he will have hemophilia?

Solution. Let Q_h be the event that the queen carries the gene for hemophilia and let Q_n be the event that she does not. Let $P_{i,h}$ be the event that the i-th prince does not have hemophilia and let $P_{i,n}$ be the event that he does not. Then

$$P(Q_h) = \frac{1}{2}, \quad P(P_{i,h}|Q_h) = \frac{1}{2}, \quad P(P_{i,h}|Q_n) = 0.$$

This implies that

$$P(Q_n) = 1 - P(Q_h) = \frac{1}{2}, \quad P(P_{i,n}|Q_h) = 1 - P(P_{i,h}|Q_h) = \frac{1}{2}, \quad P(P_{i,n}|Q_n) = 1 - P(P_{i,h}) = 1.$$

We assume that the events $P_{i,h},\ i = 1, 2, \ldots$ *are* **conditionally independent given** Q_h. We also assume that if the queen is not carrying the gene for hemophilia then no prince will get it.

Applying Bayes' formula we obtain

$$\begin{aligned}
&P(Q_h|P_{1,n} \cap P_{2,n} \cap P_{3,n}) \\
&\qquad = \frac{P(P_{1,n} \cap P_{2,n} \cap P_{3,n}|Q_h)P(Q_h)}{P(P_{1,n} \cap P_{2,n} \cap P_{3,n}|Q_h)P(Q_h) + P(P_{1,n} \cap P_{2,n} \cap P_{3,n}|Q_n)P(Q_n)} \\
&\qquad = \frac{P(P_{1,n}|Q_h)P(P_{2,n}|Q_h)P(P_{3,n}|Q_h)P(Q_h)}{P(P_{1,n}|Q_h)P(P_{2,n}|Q_h)P(P_{3,n}|Q_h)P(Q_h) + P(P_{1,n} \cap P_{2,n} \cap P_{3,n}|Q_n)P(Q_n)} \\
&\qquad = \frac{(\frac{1}{2})^4}{(\frac{1}{2})^4 + 1\,\frac{1}{2}} \\
&\qquad = \frac{1}{9}.
\end{aligned}$$

We also have

$$\begin{aligned}
&P(P_{1,n} \cap P_{2,n} \cap P_{3,n} \cap P_{4,h}) \\
&\qquad = P(P_{1,n} \cap P_{2,n} \cap P_{3,n} \cap P_{4,h}|Q_h)P(Q_h) + P(P_{1,n} \cap P_{2,n} \cap P_{3,n} \cap P_{4,h}|Q_n)P(Q_n) \\
&\qquad = (\frac{1}{2})^5 + 0 \\
&\qquad = \frac{1}{32}
\end{aligned}$$

and

$$\begin{aligned}
&P(P_{1,n} \cap P_{2,n} \cap P_{3,n}) \\
&\qquad = P(P_{1,n} \cap P_{2,n} \cap P_{3,n}|Q_h)P(Q_h) + P(P_{1,n} \cap P_{2,n} \cap P_{3,n}|Q_n)P(Q_n) \\
&\qquad = (\frac{1}{2})^4 + \frac{1}{2} = \frac{9}{16}
\end{aligned}$$

so

$$\begin{aligned}
&P(P_{4,h}|P_{1,n} \cap P_{2,n} \cap P_{3,n}) \\
&\qquad = \frac{P(P_{1,n} \cap P_{2,n} \cap P_{3,n} \cap P_{4,h})}{P(P_{1,n} \cap P_{2,n} \cap P_{3,n})} \\
&\qquad = \frac{\frac{1}{32}}{\frac{9}{16}} \\
&\qquad = \frac{1}{18}.
\end{aligned}$$

p. 106, n. 20. An urn initially contain 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replace in the urn along with 2 other balls of the same color. Compute the probability that

(a) the first 2 balls selected are black and the next 2 white.

(b) of the first 4 balls selected, exactly 2 are black.

For (a) we have

$$\begin{aligned}
&P(W_4 \cap W_3 \cap B_2 \cap B_1) \\
&\qquad = P(W_4|W_3 \cap B_2 \cap B_1)P(W_3|B_2 \cap B_1)P(B_2|B_1)P(B_1) \\
&\qquad = \frac{7}{18}\frac{5}{16}\frac{9}{14}\frac{7}{12} \\
&\qquad = \frac{35}{768} \\
&\qquad = .0456.
\end{aligned}$$

The answer to (b) is 6 times the answer to (a). This is because there are 6 two element subsets of $\{1, 2, 3, 4\}$ which can indicate on which draw the two black balls occur. For each of these six possibilities the probability will be as in (a) because the denominators will be the same and there will always be two 5's and two 7's in the numerator.

p. 110, n. 58. Barbara and Dianne go target shooting. Suppose that each of Barbara's shots hits the wooden duck target with probability p_b, while each shot of Diannes's hits it with probability p_d. Suppose that they shoot simultaneously at the same target. If the wooden duck is knocked over (indicating that it was hit), what is the probability that

(a) both shots his the duck;

(b) Barbara's shot hit the duck?

What independence assumptions have you made?

Let B_h, B_m, D_h, D_m be the probability that Barbara hits, Barbara misses, Dianne hits, Dianne misses, respectively. Then

$$P(B_h) = p_b, \quad P(B_m) = 1 - p_b, \quad P(D_h) = p_d, \quad P(B_m) = 1 - p_d.$$

We assume that B_h and D_h are independent events; this is equivalent to assuming that each of the events B_h, B_m is independent of each of the events D_h, D_m. Thus

$$P(B_h \cup D_h) = P(B_h) + P(D_h) - P(B_h \cap D_h) = P(B_h) + P(D_h) - P(B_h)P(D_h) = p_b + p_d - p_b p_d.$$

For the first part we compute

$$P(B_h \cap D_h | B_h \cup D_h) = \frac{P(B_h \cap D_h)}{P(B_h \cup D_h)} = \frac{p_b p_d}{p_b + p_d - p_b p_d}.$$

For the second part we compute

$$P(B_h | B_h \cup D_h) = \frac{P(B_h)}{P(B_h \cup D_h)} = \frac{p_b}{p_b + p_d - p_b p_d}.$$

p. 119, n. 1. In a game of bridge, West has no aces. What is the probability of his partner's having (a) no aces. (b) 2 or more aces? (c) What would the probabilities be if West had exactly 1 ace?

Condition on what West has. Of the remaining 39 cards, four are aces; thus the answer to (a) is

$$\frac{\binom{35}{13}}{\binom{39}{13}}.$$

The remaining parts are similar.

p. 108, n. 38. There cooks, A, B and C, bake a special kind of cake, and with respective probabilities .02, .03 and .05 it fails to rise. In the restaurant where they work, A bakes 50 percent of these cakes, B 30 percent, and C 20 percent. What proportion of "failures" is caused by A?

Stuff into Bayes' formula. Let F be the event that the cake fails to rise. Then

$$P(A|F) = \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C)} = \frac{.02\,.50.}{02\,.50 + .03\,.30 + .05\,.20} \simeq .3448.$$