The Poisson Distribution.

Suppose R is a region in $\mathbf{R}^n$. Let $\mathcal{L}$ be the family of Legesgue measurable subsets of R with finite $n-$area. Suppose for each A in $\mathcal{L}$ there is a random variable

$$N_A$$

with the following properties:

(i) The range of N_A is a is a subset of $\{0, 1, 2, \ldots\}$;

(ii) The cdf of N_A depends only on the $n-$area $|A|$ of A;

(iii) If $A_1, \ldots, A_m$ is a disjointed sequence of sets in $\mathcal{L}$ then

$$\{N_{A_1}, \ldots, N_{A_m}\} \quad \text{are independent and} \quad N_{\cup_{i=1}^m A_i} = \sum_{i=1}^m N_{A_i}.$$

Theorem. There is $\lambda > 0$ with the property that

$$P(N_A = k) = e^{\lambda|A|}(\lambda|A|)^k/k! \quad \text{for } k = 0, 1, 2, \ldots.$$

Sketch of proof. For each $k = 0, 1, 2, \ldots$ define

$$p_k : [0, |R|] \to [0, 1]$$

by requiring that

$$p_k(t) = P(N_A = k)$$

whenever $0 \le t < \infty$ and $A \in \mathcal{L}$ is such that $|A| = t$.

The basic observation is that

$$p_k(t + h) = \sum_{j=0}^{k} p_j(t)p_{k-j}(h)$$

for $k = 0, 1, 2, ldots$ and $t, h > 0$. To see this, choose $A, B \in \mathcal{L}$ such $|A| = t$ and $|B| = h$ and note that

$$P(N_{A\cup B} = k) = P(N_A + N_B) = \sum_{j=0}^{n} P(N_A = j, N_B = k - j) = \sum_{j=0}^{n} P(N_A = j)P(N_B = k - j),$$

which is just what we want. Some further argument is necessary to reach the desired formulae for the p_k's. Note, for example, that p_0 is nonincreasing and that

$$p_0(t + h) = p_0(t)p_0(h) \quad \text{for } t, h > 0.$$

This readily implies that, for some $\lambda > 0$,

$$p_0(t) = e^{-\lambda t} \quad \text{for all } t > 0.$$

And so forth.