My version of n. 58 on page 295.

I made a mistake when I did this in class!

Suppose X, Y, Z are indepenent random variables each of which is exponentially distributed with parameter $\lambda = 1$. Thus

$$f_{X,Y,Z}(x,y,z) = \begin{cases} e^{-(x+y+z)} & \text{if } x > 0,\ y > 0,\ z > 0, \\ 0 & \text{else.} \end{cases}$$

Let

$$U = Y + Z, \quad V = X + Z, \quad W = X + Y.$$

We shall compute the joint pdf of (U, V, W).

For each $(x, y, z) \in \mathbf{R}^3$ let

$$u(x,y,z) = y + z, \quad v(x,y,z) = x + z, \quad w(x,y,z) = x + y$$

and let

$$\mathbf{G}(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z)) = (y + z, x + z, x + y).$$

Evidently,

$$(U, V, W) = \mathbf{G}(X, Y, Y) = (u(X,Y,Z), v(X,Y,Z), w(X,Y,Z)).$$

Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Note that

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \tag{1}$$

and that

$$\mathbf{det}\, A = 2. \tag{2}$$

Whenever (x, y, z) and (u, v, w) are in $\mathbf{R}^3$ we infer from (1) that

$$(u, v, w) = \mathbf{G}(x, y, z) \ \Leftrightarrow\ \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \Leftrightarrow\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{3}$$

Equivalently,

$$\begin{matrix} u = y + z \\ v = x + z \\ w = x + y \end{matrix} \ \Leftrightarrow\ \begin{matrix} x = \frac{1}{2}(-u + v + w) \\ y = \frac{1}{2}(u - v + w) \\ z = \frac{1}{2}(u + v - w) \end{matrix} \qquad \text{whenever } (u, v, w), (x, y, z) \in \mathbf{R}^3. \tag{4}$$

Let

$$P = \{(x,y,z) : x > 0,\ y > 0,\ z > 0\} \quad \text{and let} \quad Q = \{\mathbf{G}(x,y,z) : (x,y,z) \in P\}.$$

(My formula for Q in class was wrong.) It follows from (2) that

$$Q = \{(u,v,w) \in \mathbf{R}^3 : u < v + w,\ v < u + w,\ w < u + v\}. \tag{5}$$

Make sure you understand this. Note that (4) implies that

$$(u,v,w) \in Q \ \Rightarrow\ u > 0,\ v > 0,\ w > 0.$$

It follows from (1) that $\mathbf{G}$ carries $\mathbf{R}^3$ one-to-one onto $\mathbf{R}^3$ and it is evident that

$$\begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_z & w_y & w_z \end{bmatrix} = A. \tag{6}$$

Moreover,

$$(u,v,w) = \mathbf{G}(x,y,z) \ \Rightarrow x + y + z = \frac{1}{2}(u + v + w). \tag{7}$$

Combining the above with the transformation formula in the book we obtain

$$\begin{aligned} f_{U,V,W}(u,v,w) &= \begin{cases} \frac{f_{X,Y,Z}(x,y,z)}{|\mathbf{det}\,A|} & \text{if } (u,v,w) \in Q \text{ and } (u,v,w) = \mathbf{G}(x,y,z), \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{2}e^{-(x+y+z)} & \text{if } (u,v,w) \in Q \text{ and } (u,v,w) = \mathbf{G}(x,y,z), \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{2}e^{\frac{-u+v+w}{2}} & \text{if } 0 < u < v + w,\ 0 < v < u + w,\ 0 < w < u + v, \\ 0 & \text{else} \end{cases}. \end{aligned}$$