Homework Five.

p.275 n.2. Suppose X has density $f(x) = c/x^4$ for $x > 1$, and $f(x) = 0$ otherwise, where c is a constant. Find a) c; b) $E(X)$; c) $\mathrm{Var}(X)$.

We have

$$1 = \int_1^\infty \frac{c}{x^4}\,dx = c\left[\frac{x^{-3}}{-3}\right]_1^\infty = \frac{c}{3}$$

so $c = 3$. Thus

$$E(X) = \int_{-\infty}^\infty x f(x)\,dx = \int_1^\infty x \frac{3}{x^4}\,dx = \frac{3}{2}$$

and

$$E(X^2) = \int_{-\infty}^\infty x^2 f(x)\,dx = \int_1^\infty x^2 \frac{3}{x^4}\,dx = 3$$

so

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{3}{4}.$$

p.275 n.3. Suppose X is a random variable whose density if $f(x) = cx(1-x)$ for $0 < x < 1$, and $f(x) = 0$ otherwise. Find: a) the value of c; b) $P(X \le 1/2)$; c) $P(X \le 1/3)$.

We have

$$1 = \int_0^1 cx(1-x)\,dx = \frac{c}{6}$$

so $c = 6$. We have

$$P(X \le 1/2) = \int_{x \le 1/2} f(x)\,dx = \int_0^{1/2} 6x(1-x)\,dx = \frac{1}{2}$$

and

$$P(X \le 1/3) = \int_{x \le 1/3} f(x)\,dx = \int_0^{1/3} 6x(1-x)\,dx = \frac{7}{27}.$$

p.276 n.6. Suppose X has the normal (μ, σ^2) distribution, and $P(X \le 0) = 1/3$, $P(X \le 1) = 2/3$. a) What are the values of μ and σ? b) What if $P(X \le 1) = 3/4$?

Whenever $-\infty < a < b < \infty$ we have

$$P(a < X \le b) = P\left(\frac{a-\mu}{\sigma} < Z \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

(Make sure you understand why!) Now

$$\begin{aligned} P(X \leq 0) &\leq 1/3 \\ &\Leftrightarrow \Phi(-\mu/\sigma) = 1/3 \\ &\Leftrightarrow \Phi(\mu/\sigma) = 1 - 1/3 = 2/3 \\ &\Leftrightarrow \mu/\sigma = \Phi^{-1}(2/3) \\ &\Rightarrow \mu/\sigma \approx \frac{2/3 - .6736}{.6772 - .6736}\Phi^{-1}(.6736) + \frac{.6772 - 2/3}{.6772 - .6736}\Phi^{-1}(.6772) \\ &\Rightarrow \mu/\sigma \approx \frac{2/3 - .6736}{.6772 - .6736}.45 + \frac{.6772 - 2/3}{.6772 - .6736}.46 \\ &\Rightarrow \mu/\sigma \approx 0.4792; \end{aligned}$$

$$\begin{aligned} P(X \leq 1) &\leq 2/3 \\ &\Leftrightarrow (1-\mu)/\sigma = \Phi^{-1}(2/3) \\ &\Rightarrow (1-\mu)/\sigma \approx .4792; \quad \text{(See above.)} \end{aligned}$$

$$\begin{aligned} P(X \leq 1) &\leq 3/4 \\ &\Leftrightarrow \Phi((1-\mu)/\sigma) = 3/4 \\ &\Leftrightarrow (1-\mu)/\sigma = \Phi^{-1}(3/4) \\ &\Rightarrow (1-\mu)/\sigma \approx \frac{3/4 - .7486}{.7517 - .7486}\Phi^{-1}(.7486) + \frac{.7517 - 3/4}{.7517 - .7486}\Phi^{-1}(.7517) \\ &\Rightarrow (1-\mu)/\sigma \approx \frac{3/4 - .7486}{.7517 - .7486}.67 + \frac{.7517 - 3/4}{.7517 - .7486}.68 \\ &\Rightarrow (1-\mu)/\sigma \approx .0.6754. \end{aligned}$$

Thus if $P(X \leq 0) = 1/3$ and $P(X \leq 1) = 2/3$ we have we have

$$\mu/\sigma \approx .4792 \quad \text{and} \quad (1-\mu)/\sigma \approx .4792$$

so

$$\mu \approx .500 \quad \text{and} \quad \sigma \approx 1.043.$$

and if If $P(X \leq 0) = 1/3$ and $P(X \leq 1) = 3/4$ we have

$$\mu/\sigma \approx .4792 \quad \text{and} \quad (\mu - 1)/\sigma \approx .6754$$

so

$$\mu = 0.4150 \quad \text{and} \quad \sigma \approx 0.866.$$

p.276 n.12. The point here is that if (X, Y) is uniformly distributed on the region D in $\mathbb{R}^2$ then

$$f_X(x) = \lim_{h \downarrow 0} \frac{1}{h} \frac{|D \cap ((x, x+h) \times \mathbb{R})|}{|D|} = \frac{\text{length of}\{y \in \mathbb{R} : (x.y) \in D\}}{|D|}.$$

Thus for a) we have

$$f_X(x) = \frac{1}{8} \begin{cases} 0 & \text{if } x \leq -2; \\ (x+2) - (-x-2) = 2(x+2) & \text{if } -2 < x \leq 0; \\ (-x+2) - (x-2) = 2(2-x) & \text{if } 0 < x \leq 2; \\ 0 & \text{if } 2 < x. \end{cases}$$

For b) we have

$$f_X(x) = \frac{1}{3}\begin{cases} 0 & \text{if } x \le -2; \\ x+2 & \text{if } -2 < x \le 0; \\ -2x+2 = 2(1-x) & \text{if } 0 < x \le 2; \\ 0 & \text{if } 2 < x. \end{cases}$$

For c) we have

$$f_X(x) = \frac{1}{1}\begin{cases} 0 & \text{if } x \le -1; \\ (2x+2) - (x/2+1/2) & \text{if } -1 < x \le 0; \\ (x/2+2) - (x/2+1/2) & \text{if } 0 < x \le 1; \\ (x/2+2) - (2x-3) & \text{if } 1 < x \le 2; \\ 0 & \text{if } 2 < x. \end{cases}$$

p.277 n.13. Let X be the length of a rod produced by the process. Then

$$f_X(x) = c\begin{cases} x - .9 & \text{if } .9 < x < 1.0, \\ 1.1 - x & \text{if } 1 \le x < 1.1. \end{cases}$$

Since

$$\int_{-\infty}^{\infty} f_X(x)\,dx = .01$$

we find that the integral of f_X must be 1 we find that

$$c = 100.$$

The answer to a) is

$$P(.925 < X < 1.025) = 100\int_{.925}^{1.025} f_X(x)\,dx = .9375;$$

This can be also be seen by considering areas of triangles; in fact, that's a better way to do it.

For b), let

$$p = P(.95 < X < 1.05 | .925 < X < 1.075);$$

p is the probability that a rod delivered to the client meets the client's requirements and $q = 1 - p$ is the probability that a rod delivered to the client does not meet the client's requirements.

Since

$$P(.95 < X < 1.05) = .75$$

we find that

$$p = \frac{.75}{.9375} = \frac{4}{5}.$$

In class we showed the answer to a) is $1/4$. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent random variables with values 0 or 1 such that

$$P(X = x) = \begin{cases} p & \text{if } x = 1, \\ q & \text{if } x = 0, \\ 0 & \text{else.} \end{cases}$$

The answer to b) is the smallest n such that

$$P\left(\sum_{i=1}^{n} X_i > 99.5\right) \geq .95.$$

Now by the Central Limit Theorem

$$P\left(\sum_{i=1}^{n} X_i > 99.5\right) = P\left(\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{npq}} > \frac{99.5 - np}{\sqrt{npq}}\right) \approx 1 - \Phi\left(\frac{99.5 - np}{\sqrt{npq}}\right)$$

so we want to determine n (not necessarily an integer) so that

$$1 - \Phi\left(\frac{99.5 - np}{\sqrt{npq}}\right) = .95 \quad \text{or} \quad \Phi\left(\frac{np - 99.5}{\sqrt{npq}}\right) = .95.$$

Now from the table $\Phi(z) = .95$ if $z \approx 1.645$. Now

$$\frac{np - 99.5}{\sqrt{npq}} = 1.645$$

if $n \approx 133.9$ so we should take $n = 134$.

p.293 n.1. Suppose T is exponential with parameter λ. Then

$$P(T > t) = e^{-\lambda t}, \ 0 < t < \infty.$$

We are given that

$$P(T > 1) = \frac{1}{2}.$$

Thus $e^{-\lambda} = \frac{1}{2}$ so $\lambda = \ln 2$. For a) we have

$$P(T > 5) = e^{-\lambda 5} = e^{-(\ln 2)5} = \frac{1}{32}.$$

For b) we want t such that $P(T > t) = 1/10$ so

$$e^{-\lambda t} = \frac{1}{10}$$

so $-(\ln 2)t = -\ln 10$ so

$$t = \frac{\ln 10}{\ln 2}.$$

For c) we want t such that $P(T > t) = 1/1024$; proceeding as we did in b) we find that

$$t = \frac{\ln 1024}{\ln 2} = 10.$$

For d), we first observe that

$$e^{-\lambda \cdot 10} = \frac{1}{1024}.$$

Let X_i, $i = 1, \ldots, 1024$ be independent exponential with parameter λ. The answer to d) is

$$\begin{aligned} P(X_i \leq 10, \ i = 1, \ldots, 1024) &= (P(X_i \leq 10)^{1024} \\ &= (1 - P(X_i > 10))^{1024} \\ &= (1 - e^{-\lambda \cdot 10})^{1024} \end{aligned}$$

the natural log of which is $1024(\ln 1023 - \ln 1024)$ so that answer to d) is the exponential of this which is

$$\approx 0.3677.$$

p.293 n.4. Let T be exponential with parameter $\lambda = 1/10$ so $E(T) = 1/\lambda = 10$.

For a) we want $P(T > 20) = e^{-\lambda(20)} = e^{-2}$. For b) we want t such that $F_T(t) = 1/2$ which is to say that

$$1 - e^{-\lambda t} = \frac{1}{2}$$

or $e^{-t/10} = 1/2$ or $-t/10 = -\ln 2$ or $t = (\ln 2)/10$. For c) the SD of T is $1/\lambda = 10$. For d) we want

$$P(\frac{1}{100}\sum_{i=1}^{100} T_i > 11)$$

where T_i, $i = 1, \ldots, 100$ are independent exponential with parameter λ. Thus $\sum_{i=1}^{100} T_i$ has the Gamma distribution with parameter $n = 100$ and $\lambda = 10$. Now the density for this distribution is (see p.286)

$$(0, \infty) \ni t \mapsto \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}.$$

so the desired probability is, courtesy of Maple, (but see (2) in the box on page 286),

$$\int_{1100}^{\infty} (1/10)e^{-(1/10)t}\frac{((1/10)t)^{100-1}}{(100-1)!} \approx 0.1582786700.$$

For e) we replace 100 by 2 getting 0.3545701069.

Notice I computed *probabilities* in d) and e). The book has the phrase "probability that the average lifetime of..". This make no sense: an "average lifetime" is the expectation of a lifetime which assumes "lifetime" is a random variable so its expectation is a number. You can't take the probability of a number. So the word "average" should be omitted.

p.293 n.5. For each $n = 1, 2, 3, \ldots$ let T_n be the time of arrival in seconds of the nth call. Then for any positive integer n we have that T_n has the Gamma distribution with parameters n and $\lambda = 1$ and that

$$T_1, T_2 - T_1, \ldots, T_n - T_{n-1}$$

are independent exponential with parameter $\lambda = 1$.

For a) we have

$$P(T_4 - T_3 \leq 2) = 1 - e^{-2} \approx 0.8646647168.$$

Now T_4 has the Gamma distribution with parameters $n = 4$ and $\lambda = 1$.. So for b) we have, courtesy of Maple,

$$P(T_4 \leq 5) = \int_0^5 e^{-x}\frac{x^{4-1}}{(4-1)!}\,dx = 6 - 236e^{-5} \approx 0.7349740847$$

and for c) we have (see p.286)

$$E(T_4) = \frac{4}{1} = .4$$

p.294 n.9. For a) we have

$$\begin{aligned}\Gamma(r+1) &= \int_0^\infty x^r\, e^{-x}\, dx \\ &= -\int_0^\infty x^r\, d(e^{-x}) \\ &= -x^r\, e^{-x}\Big|_0^\infty + \int_0^\infty d(x^r)\, e^{-x}\, dx \\ &= r\int_0^\infty x^{r-1}\, e^{-x}\, dx \\ &= r\Gamma(r).\end{aligned}$$

For b) we have

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1 = 0!.$$

So

$$\Gamma(r) = (r-1)! \;\Rightarrow\; \Gamma(r+1) = r\Gamma(r) = r(r-1)! = r!.$$

For c) we have from b)

$$E(T^n) = \int_0^\infty x^n f_T(x)\, dx = \int_0^\infty x^n\, e^{-x}\, dx = \Gamma(n+1) = n!.$$

For d) first recall that if X is a random variable and a is a positive constant then

$$E(aX) = aE(x) \text{ and } \mathrm{Var}(aX) = a^2\mathrm{Var}(X).$$

Now

$$F_{\lambda T}(u) = P(\lambda T \le u) = P(T \le u/\lambda) = 1 - e^{-\lambda(u/\lambda} = 1 - e^{-u}$$

so λT is exponential with parameter 1. From p.279 we have that

$$E(\lambda T) = \mathrm{Var}(\lambda T) = 1.$$

From c) we obtain

$$\lambda^n E(T^n) = E((\lambda T)^n) = n!$$

so

$$E(T^n) = \frac{n!}{\lambda^n}.$$

Moreover,

$$\lambda^2\mathrm{Var}(T) = \mathrm{Var}(\lambda T) = 1$$

so

$$\mathrm{Var}(T) = \frac{1}{\lambda}.$$

(But this is on p.279 also; so this part of the problem is screwed up.)

p.295 n.14. Suppose $\lambda = .05$ and let T be exponential with parameter λ. Then

$$P(T<1) = P(T \le 1) = F_T(1) = 1 - e^{-\lambda} = \lambda - \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!} + \cdots.$$

so the answer is "slightly less than .05".