The multinomial distribution.

Let n and r be positive integers and let

$$p_1, \ldots, p_r$$

be such that

$$0 \le p_\alpha \le 1 \quad \text{and} \quad \sum_{\alpha=1}^{r} p_\alpha = 1.$$

Let

$$\mathbf{M}(n, r)$$

be the set of r-tuples $(n_1, \ldots, n_r)$ of nonnegative integers such that $\sum_{\alpha=1}^{r} = n$. The discrete random vector $(N_1, \ldots, N_r)$ is said to have the r**-nomial (multinomial in the book) distribution with parameters** n **and** $p_1, \ldots, p_r$ if

$$p_{N_1,\ldots,N_n}(n_1, \ldots, n_r) = \begin{cases} \binom{n}{n_1 \cdots n_r} p_1^{n_1} \cdots p_r^{n_r} & \text{if } (n_1, \ldots, n_r) \in \mathbf{M}(n, r), \\ 0 & \text{else.} \end{cases}$$

Note that

$$1 = (p_1 + \cdots + p_r)^n = \sum_{(n_1,\ldots,n_r) \in \mathbf{M}(n,r)} \binom{n}{n_1 \cdots n_r} p_1^{n_1} \cdots p_r^{n_r}$$

by the multinomial theorem.

Example. An urn contain 3 red balls, 4 white balls and 5 blue balls. A ball is drawn from the urn 10 times with replacement. Let R, W and B be the number of red,white and blue balls drawn, respectively. Then, as we shall see below, (R, W, B) will have the 3-nomial distribution with parameters $n = 10$ and

$$p_1 = \frac{3}{3+4+5}, \quad p_2 = \frac{4}{3+4+5}, \quad p_2 = \frac{5}{3+4+5}.$$

Note that the range of (R, B, W) is the set of 3-tuples (n_r, n_w, n_b) of nonnegative integers such that $n_r + n_w + n_b = 10$.

Here is how such a random vector arises. Let

$$O = \{o_1, \ldots, o_r\}$$

be a set containing exactly r elements; O is the set of *outcomes*. Let

$$X_i, \; i = 1, \ldots, n,$$

be independent identically distributed random variables with the same range O such that

$$P(X_i = o_\alpha) = p_\alpha \quad \text{whenever } i \in \{1, \ldots, n\} \text{ and } \alpha \in \{1, \ldots, r\}.$$

For each $\alpha \in \{1, \ldots, r\}$ let

$$N_\alpha = \sum_{i=1}^{n} 1_{\{X_i = o_\alpha\}};$$

thus N_α is the number of times the α-th outcome o_α occurs in n tries. Evidently,

$$\sum_{\alpha=1}^{r} N_\alpha = n.$$

Note that N_α, being the sum of n independent Bernoulli variables with parameter p_α, is binomial with parameters $n = n$ and $p = p_\alpha$. We let

$$\mathbf{A}(n,r)$$

be the family of ordered r-tuples $(A_1, \ldots, A_r)$ of subsets of $\{1, \ldots, n\}$ such that $A_i \cap A_j = \emptyset$ and, for each $A \in \mathbf{A}(n,r)$, we let

$$E_A = \cap_{\alpha=1}^r \cap_{i \in A_\alpha} \{X_i = o_\alpha\}$$

and observe that, by the independence of the X_i,

$$P(E_A) = \prod_{\alpha=1}^r p_\alpha^{|A_\alpha|}. \tag{2}$$

If $(n_1, \ldots, n_r)$ is an r-tuple of nonnegative integers summing to n we find that

$$P(N_1 = n_1, \ldots, N_r = n_r) = P(\cup_{A \in \mathbf{A}(n,r), |A_\alpha| = n_\alpha,\ \alpha=1,\ldots,r} E_A) = \binom{n}{n_1 \cdots n_r} p_1^{n_1} \cdots p_r^{n_r}$$

so (1) holds.

Theorem. Suppose q is a integer, $1 < q < r$, $\tilde{p} = \sum_{\alpha=1}^q p_\alpha$ and

$$\tilde{p}_\alpha = \frac{p_\alpha}{\tilde{p}} \quad \text{whenever } \alpha = 1, \ldots, q.$$

Then

$$P(N_1 = n_1, \ldots, N_q = n_q | N_{q+1} = n_{q+1}, \ldots, N_r = n_r) = \binom{\sum_{\alpha=1}^q n_\alpha}{n_1 \cdots n_q} \tilde{p}_1^{n_1} \cdots \tilde{p}_q^{n_q}$$

whenever $(n_1, \ldots, n_r) \in \mathbf{M}(n,r)$.

Remark. This says that the conditional distribution of $(N_1, \ldots, N_q)$ given $\{N_{q+1} = n_{q+1}, \ldots, N_r = n_r\}$ is q-nomial with parameters $n - \sum_{\alpha=q+1}^r n_\alpha$ and $\tilde{p}_1, \ldots, \tilde{p}_q$ whenever $(n_{q+1}, \ldots, n_r)$ is a $(r-q)$-tuple of nonnegative integers whose sum does not exceed n. In particular, if $\alpha \in \{1, \ldots, q\}$ we find that the conditional distribution of N_α given $\{N_{q+1} = n_{q+1}, \ldots, N_r = n_r\}$ is binomial with parameters $n = n - \sum_{\alpha=q+1}^r n_\alpha$ and $p = \tilde{p}_\alpha$.

Proof. Suppose $(n_1, \ldots, n_r) \in \mathbf{M}(n,r)$. Let

$$E = \cap_{\alpha=1}^q \{N_\alpha = n_\alpha\} \quad \text{and let} \quad F = \cap_{\beta=q+1}^r \{N_\beta = n_\beta\}.$$

Let $\tilde{n} = \sum_{\alpha=1}^q n_\alpha$.

Let $\mathbf{B}$ be the family of q-tuples $(B_1, \ldots, B_q)$ of subsets of $\{1, \ldots, n\}$ such that $|B_\alpha| = n_\alpha$ whenever $\alpha \in \{1, \ldots, q\}$ and $B_\alpha \cap B_\beta = \emptyset$ whenever $\alpha, \beta \in \{1, \ldots, q\}$ and $\alpha \neq \beta$. For each $B \in \mathbf{B}$ let

$$E_B = \cap_{\alpha=1}^q \cap_{i \in B_\alpha}^n \{X_i = o_\alpha\}.$$

Let $\mathbf{B}'$ be the set of those $(B_1, \ldots, B_q) \in \mathbf{B}$ such that $B_\alpha \subset \{1, \ldots, \tilde{n}\}$ whenever $\alpha \in \{1, \ldots, q\}$.

Let $\mathbf{C}$ be the family of $(r-q)$-tuples $(C_{q+1}, \ldots, C_r)$ of subsets of $\{1, \ldots, n\}$ such that $|C_\beta| = n_\beta$ whenever $\beta \in \{q+1, \ldots, r\}$ and $C_\beta \cap C_\gamma = \emptyset$ whenever $\beta, \gamma \in \{q+1, \ldots, r\}$ and $\beta \neq \gamma$. Let $\tilde{C}$ be that member of $\mathbf{B}$ such that

$$\tilde{C}_\beta = \{\sum_{\gamma=1}^{\beta-1} n_\gamma + 1, \ldots, \sum_{\gamma=1}^{\beta} n_\gamma\},\ \beta = q+1, \ldots, r.$$

For each $B \in \mathbf{B}$ let

$$F_B = \cap_{\beta=q+1}^r \cap_{i \in C_\beta}^n \{X_i = o_\beta\}.$$

One may easily verify that

$$P(F_C) = P(F_D) \quad \text{and} \quad P(E|F_C) = P(E|F_D) \quad \text{whenever } C, D \in \mathbf{C}.$$

By an earlier result we have that

$$P(E|F) = P(E|F_{\tilde{C}}).$$

Moreover,

$$\begin{aligned}
P(E|F_{\tilde{C}}) &= \sum_{(B_1,\ldots,B_q)\in\mathbf{B}} P(E_B|F_{\tilde{C}}) \\
&= \sum_{(B_1,\ldots,B_q)\in\mathbf{B}} \frac{P(E_B \cap F_{\tilde{C}})}{P(F_{\tilde{C}})} \\
&= \sum_{(B_1,\ldots,B_q)\in\mathbf{B}'} \frac{P(E_B \cap F_{\tilde{C}})}{P(F_{\tilde{C}})} \\
&= \sum_{(B_1,\ldots,B_q)\in\mathbf{B}'} \frac{{p_1}^{n_1} \ldots {p_q}^{n_q} {p_{q+1}}^{n_{q+1}} \cdots {p_r}^{n_r}}{\tilde{p}^{\tilde{n}} {p_{q+1}}^{n_{q+1}} \cdots {p_r}^{n_r}} \\
&= \binom{\tilde{n}}{n_1 \cdots n_q} \tilde{p}_1^{n_1} \cdots \tilde{p}_q^{n_q}.
\end{aligned}$$

□