More on expectation and random variables.

Definition. Suppose $(S, \mathcal{E}, P)$ is a probability space and $X : S \to \mathbf{R}$ is a random variable. Suppose

$$g : \mathbf{R} \to \mathbf{R}.$$

Let

$$g(X) : \mathbf{R} \to \mathbf{R}$$

be such that

$$g(X)(s) = g(X(s)) \quad \text{whenever } s \in S.$$

Thus $g(X)$ is $g \circ X$ where $\circ$ is composition of functions. If g is a Borel function then $g(X)$ is a random variable. Any g's we deal with will be Borel function. I omit the technical definition of Borel function.

Theorem. Suppose $(S, \mathcal{E}, P)$ is a probability space, $X : S \to \mathbf{R}$ is a random variable and

$$g : \mathbf{R} \to \mathbf{R}$$

is a Borel function. Then

$$E(g(X)) = \sum_{x \in \mathbf{R}} g(x) p_X(x).$$

Proof. We prove this in the case where the range of X is finite, say $\mathbf{rng}\, X = \{x_1, \ldots, x_m\}$ where $x_1 < \cdots < x_m$. Then

$$\begin{aligned} E(g(X)) &= \sum_{s \in S} g(X(s)) P(\{s\}) \\ &= \sum_{i=1}^{m} g(x_i) \sum_{s \in \{X = x_i\}} P(\{s\}) \\ &= \sum_{i=1}^{m} g(x_i) P(X = x_i) \\ &= \sum_{i=1}^{m} g(x_i) p_X(x_i). \end{aligned}$$

□

Theorem. Suppose X and Y are independent random variables (on the same probability space). Then

$$E(XY) = E(X)E(Y).$$

Proof. We prove this in the case where the ranges of X and Y are finite,$\mathbf{rng}\, X = \{x_1, \ldots, x_m\}$ where $x_1 < \cdots < x_m$ and $\mathbf{rng}\, Y = \{y_1, \ldots, y_n\}$ where $y_1 < \cdots < y_n$. Let $Z = XY$ and let $z_1 < \cdots < z_p$ be such that the range of Z is $\{z_1, \ldots, z_p\}$.

For each $k = 1, \ldots, p$ let $E_k = \{(x_i, j_j) : x_j y_k = z_k\}$, $k = 1, \ldots, p$. Then

$$\begin{aligned} P(Z = z_k) &= P(XY = z_k) \\ &= P(\cup_{(x_i, y_j) \in E_k \{X = x_i|} \cap \{Y = y_j\}) \\ &= \sum_{((x_i, y_j) \in E_k} P(\{X = x_i\} \cap \{Y = y_j\}) \\ &= \sum_{((x_i, y_j) \in E_k} P(\{X = x_i\}) P(\{Y = y_j\}); \end{aligned}$$

the last step is valid because X and Y are independent.

Then

$$E(Z) = \sum_{k=1}^{p} z_k P(Z = z_k)$$

$$= \sum_{k=1}^{p} \sum_{(x_i, y_j) \in E_k} x_i y_j P(X = x_i) P(Y = y_j)$$

$$= \big(\sum_{i=1}^{m} x_i P(X = x_i) \big) \big(\sum_{j=1}^{n} y_j P(Y = y_j) \big)$$

$$= E(X)E(Y).$$

□