

**Some basic facts about integration.**

**The case  $\mathbf{R}^2$ .**

Suppose  $R \subset \mathbf{R}^2$ . Then

$$(1) \quad \int \int_R f(x, y) dx dy = \int_{\mathbf{R}} \left( \int_{\{y:(x,y) \in R\}} f(x, y) dy \right) dx$$

In particular, if  $f$  is the indicator function of  $R$  we find that

$$(2) \quad \text{Area}(R) = \int_{\mathbf{R}} \text{Length}(\{y : (x, y) \in R\}) dx.$$

An important special case is the following. Suppose  $-\infty < a < b < \infty$ ;  $\alpha, \beta : (a, b) \rightarrow \mathbf{R}$ ;

$$(3) \quad \alpha(x) < \beta(x) \quad \text{whenever } x \in (a, b);$$

and

$$R = \{(x, y) \in \mathbf{R}^2 : x \in (a, b) \text{ and } \alpha(x) < y < \beta(x)\}.$$

Then

$$\{y : (x, y) \in R\} = \begin{cases} (\alpha(x), \beta(x)) & \text{if } a < x < b, \\ \emptyset & \text{else} \end{cases}$$

so we obtain

$$(4) \quad \int \int_R f(x, y) dx dy = \int_a^b \left( \int_{\alpha(x)}^{\beta(x)} f(x, y) dy \right) dx$$

and

$$(5) \quad \text{Area}(R) = \int_a^b \beta(x) - \alpha(x) dx.$$

Note that this all breaks down if (3) does not hold.

Note that in the the foregoing  $(x, y)$  may be replaced by any rectangular coordinate system on  $\mathbf{R}^2$ ; in particular, we may interchange  $x$  and  $y$ .

**The case  $\mathbf{R}^3$ .**

Suppose  $R \subset \mathbf{R}^3$ .

Then

$$(1) \quad \begin{aligned} \int \int \int_R f(x, y, z) dx dy dz &= \int_{\mathbf{R}} \left( \int \int_{\{(y,z):(x,y,z) \in R\}} f(x, y, z) dy dz \right) dx \\ &= \int_{\mathbf{R}} \left( \int_{\mathbf{R}} \left( \int_{\{z:(x,y,z) \in R\}} f(x, y, z) dz \right) dy \right) dx \\ &= \int \int_{\mathbf{R}^2} \left( \int_{\{z:(x,y,z) \in R\}} f(x, y, z) dz \right) dx dy. \end{aligned}$$

In particular, if  $f$  is the indicator function of  $R$  we find that

$$(3) \quad \text{Volume}(R) = \int_{\mathbf{R}} \text{Area}(\{(y, z) : (x, y, z) \in R\}) dx$$

and

$$(4) \quad \text{Volume}(R) = \int_{\mathbf{R}^2} \text{Length}(\{z : (x, y, z) \in R\}) dx dy$$

An important special case is the following. Suppose  $-\infty < a < b < \infty$ ;  $\alpha, \beta : (a, b) \rightarrow \mathbf{R}$ ;

$$(5) \quad \begin{aligned} \alpha(x) &< \beta(x) \quad \text{whenever } x \in (a, b); \\ Q &= \{(x, y) \in \mathbf{R}^2 : a < x < b \text{ and } \alpha(x) < y < \beta(x)\}; \end{aligned}$$

$$\gamma, \delta : Q \rightarrow \mathbf{R};$$

$$(6) \quad \gamma(x, y) < \delta(x, y) \quad \text{whenever } (x, y) \in Q;$$

and

$$R = \{(x, y, z) \in \mathbf{R}^3 : x \in (a, b), \alpha(x) < y < \beta(x) \text{ and } \gamma(x, y) < z < \delta(x, y)\}.$$

Then

$$(7) \quad \int \int \int_R f(x, y, z) dx dy dz = \int_a^b \left( \int_{\alpha(x)}^{\beta(x)} \left( \int_{\gamma(x, y)}^{\delta(x, y)} f(x, y, z) dz \right) dy \right) dx.$$

and

$$(8) \quad \text{Volume}(R) = \int_a^b \left( \int_{\alpha(x)}^{\beta(x)} \delta(x, y) - \gamma(x, y) dy \right) dx.$$

Note that this all breaks down if either (5) or (6) does not hold.

Note that  $(x, y, z)$  may be replaced by any rectilinear coordinate system on  $\mathbf{R}^3$ ; in particular, we may interchange  $x, y$  and  $z$ .

**Example.** Suppose  $A, B, C$  are independent uniformly distributed on  $(0, 1)$ . Compute  $P(4AC < B^2)$ .

**Solution.** Let

$$R = \{(a, b, c) : 0 < a < 1, 0 < b < 1, 0 < c < 1 \text{ and } 4ac < b^2\}.$$

Then

$$P(4AC < B^2) = \text{Volume}(R).$$

Let

$$Q = \{(a, c) : 0 < a < 1, 0 < c < 1 \text{ and } 4ac < 1\}.$$

Then

$$R = \{(a, b, c) : (a, c) \in Q \text{ and } 2\sqrt{ac} < b < 1\}$$

so, after drawing a picture of  $Q$ , we find that

$$\begin{aligned} \text{Volume}(R) &= \int \int_Q 1 - 2\sqrt{ac} da dc \\ &= \int_0^{1/4} \left( \int_0^1 1 - 2\sqrt{ac} dc \right) da + \int_{1/4}^1 \left( \int_0^{1/4a} 1 - 2\sqrt{ac} dc \right) da \\ &= \frac{5}{36} + \frac{1}{6} \ln 2. \end{aligned}$$

Alternatively, we could set

$$S_b = \{(a, c) : (a, b, c) \in R\} \quad \text{for } b \in \mathbf{R}$$

and, again after drawing a picture, calculate

$$\text{Area}(S_b) = \begin{cases} \frac{b^2}{4} (1 - \ln \frac{b^2}{4}) & \text{if } 0 < b <, \\ \emptyset & \text{else} \end{cases}$$

obtaining

$$\text{Volume}(R) = \int_0^1 \frac{b^2}{4} (1 - \ln \frac{b^2}{4}) db = \frac{5}{36} + \frac{1}{6} \ln 2.$$