Integrating the Gaussian.

For each real x we let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and we let

$$\Phi(x) = \int_{-infty}^{x} \phi(w)\, dw.$$

ϕ is called the **standard Gaussian** (I think) and Φ is called the **standard error function**.

Theorem. We have

$$\int_{-\infty}^{\infty} \phi(x)\, dx = 1.$$

One of many proofs. We have

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx \right)^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx \right) \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy \right) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx \right) e^{-\frac{y^2}{2}}\, dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\, dx \right) dy \\
&= \int\int_{\mathbf{R}^2} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\, dxdy \\
&= \int\int_{\mathbf{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy \\
&= \int\int_{(0,\infty)\times(0,2\pi)} e^{-\frac{r^2}{2}} r\, drd\theta \\
&= \int_0^{\infty} \left(\int_0^{2\pi} e^{-\frac{r^2}{2}} r\, d\theta \right) dr \\
&= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} r\, dr \\
&= 2\pi \left[-e^{\frac{-r^2}{2}} \right]_{r=0}^{r=\infty} \\
&= 2\pi.
\end{aligned}$$

Definition. We say the random variable X is **standard normal** if $F_X = \Phi$. This implies that X is continuous with $f_X = \phi$.

Theorem. Suppose X is standard normal. Then X has mean 0 and variance 1.

Proof. We have

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-infty}^{\infty} x e^{-\frac{x^2}{2}}\, dx = 0$$

because $x \mapsto xe^{-\frac{x^2}{2}}$ is odd. Also,

$$\begin{aligned}
\sqrt{2\pi}E(X^2) &= \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\,dx \\
&= -\int_{-\infty}^{\infty} x d\big(e^{-\frac{x^2}{2}}\big) \\
&= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= \sqrt{2\pi}.
\end{aligned}$$

Definition. Suppose $-\infty < \mu < \infty$ and $0 < \sigma < \infty$. We say the random variable X is **normally distibuted with parameter μ and σ^2** or **normally distributed with mean μ and variance σ** if

$$\frac{X-\mu}{\sigma}$$

is standard normal; if this is the case we immediately infer that the mean of X is μ and the variance of X is σ^2.